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PLANE WAVES IN AN ELASTIC-PLASTIC HALF-SPACE DUE TO COMBINED SURFACE PRESSURE AND SHEAR

by

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ABSTRACT

The most general case of plane wave propagation, when normal and shear stresses occur simultaneously, is considered in a material obeying the v. Mises yield condition.

The resulting non-linear differential equations have not previously been solved for any boundary value problem, except for special situations where the differential equations degenerate into linear ones. In the present paper, the stresses in a half-space, due to a uniformly distributed step load of pressure and shear on the surface, are obtained in closed form.

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LIST OF MAJOR SYMBOLS

$E=E(\phi,\overline{k})$	elliptic integral of second kind
$F=F(\phi,\overline{k})$	elliptic integral of the first kind
f	plastic potential
G	elastic shear modulus
$H(U)$	integral defined by Eq. (3-28)
$H_u(U)$	integral defined by Eq. (A-29)
$H_v(U)$	integral defined by Eq. (A-37)
$h(t)$	unit step function
$I(U)$	integral defined by Eq. (A-2)
J_1	first invariant of the stress
J_2'	second invariant of the stress deviator
K	elastic bulk modulus
k	yield stress in simple shear
$\overline{k}$, $\hat{k}$	modulus of elliptic integrals
s_{ij}, s_x, s_y, s_z	components of stress deviator
t	time
$U=\sqrt{\frac{\rho}{G}}\ \frac{x}{t}$	non-dimensional variable
$U_P=\sqrt{\frac{4+\beta}{3}}$	value of U at elastic P-front
$U_S = 1$	value of U at elastic S-front
U_1, U_2	lower and upper boundaries of dissipative regions

$\dot{u}_i$	components of par icle velocity
$\dot{u}$	x-component of particle velocity
V, V_i	characteristic velocities
$\dot{v}$	y-component of particle velocity
x, y, z	cartesian coordinates
$\beta = \frac{2(1+\nu)}{1-2\nu}$	non-dimensional material parameter
$\Delta\sigma_x$, etc.	increment at discontinuity
δ_{ij}	Kronecker delta
ϵ_{ij}	components of strain
ϵ^E_{ij} , ϵ^P_{ij}	elastic and plastic parts of the strain
$\lambda \equiv \frac{1}{G}\dot{\Lambda} > 0$	function defining energy dissipation
ν	Poisson's ratio
$\Pi = \Pi(\phi, \alpha^2, \bar{k})$	elliptic integral of third kind
ρ	density
$\sigma_{ij}, \sigma_x, \sigma_y$	components of stress
σ_o	normal stress on the surface
σ_1 , σ_2	defined by Eqs. (4-4), (4-7)
$\tau_{xy} = \tau$	shear stress
τ_o	shear stress on the surface
ϕ , $\hat{\phi}$	amplitude of elliptic integrals

I. INTRODUCTION

The basic differential equations governing wave propagation in an ideal elastic-plastic medium subject to the v. Mises yield condition are easily formulated. It is established [1]* that these equations are hyperbolic in character, and that there are two non-vanishing pairs of characteristic velocities, $\pm V_1$, $\pm V_2$. However, the actual solution of boundary value problems is complicated by two causes: in general, the differential equations are non-linear, so that the V_i are functions of the stresses, and there are, a priori unknown, moving boundaries between elastic and plastic regions. As may be seen from a review of the field, [2], the published literature contains solutions only to one-dimensional problems of plane, spherical or cylindrical pressure waves where just one of the two characteristic velocities enters the solution, and where, furthermore, the differential equations degenerate into linear ones, so that this velocity, $\pm V$, is a constant. No multi-dimensional problem, necessarily involving non-linear differential equations with two pairs of non-constant characteristics, has at present been treated.

As a step towards multi-dimensional problems the present paper considers a case in which, while one-dimensional, the differential equations are nonlinear and both characteristic velocities enter the general solution. Subjecting a half-space, Fig. 1, to a suddenly applied, uniformly distributed normal stress $\sigma_o(t)$ combined with a shear stress

* Numbers in parentheses [ ] refer to the bibliography.

$\tau_o(t)$, plane waves of pressure combined with shear are generated. For the case of step loads, $\sigma_o(t) = \sigma_o h(t), \tau_o(t) = \tau_o h(t)$, closed form solutions for the stresses and velocities can be obtained. For such loads dimensional considerations make it possible to convert the partial differential equations into a set of simultaneous ordinary differential equations which can be solved by quadratures in terms of elliptic integrals and/or simpler transcendental functions.

II. FORMULATION OF THE BASIC EQUATIONS

The basic equations governing wave propagation in an ideal elastic-plastic continuum consist of a yield condition defining the plastic state, a set of constituitive equations relating stress, strain and/or their derivatives, and the equations of motion.

Starting with the yield condition, this report will consider a material obeying the v. Mises condition

$$f \equiv J_2' - k^2 = 0 \tag{2-1}$$

where

$$J_2' = \frac{1}{2} s_{ij} s_{ij} \tag{2-2}$$

is the second invariant of the stress deviator, and $k > 0$ is the yield stress in simple shear.

For an ideal elastic-plastic material the total strain ϵ_{ij} is composed of an elastic part ϵ^E_{ij} and a plastic part ϵ^P_{ij}, so that the total strain, and strain rate are, respectively

$$\epsilon_{ij} = \epsilon^E_{ij} + \epsilon^P_{ij} \tag{2-3}$$

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^E_{ij} + \dot{\epsilon}^P_{ij} \,. \tag{2-4}$$

Employing conventional symbols, listed on pages iv and v, the elastic portion of the strain rate is

$$\dot{\epsilon}^E_{ij} = \frac{1}{2G}\,\dot{s}_{ij} + \frac{1-2\nu}{6G(1+\nu)}\,\dot{J}_1\delta_{ij} \qquad (2\text{-}5)$$

while the plastic portion, obtained from the concept of a plastic potential [3] is

$$\dot{\epsilon}^P_{ij} = \lambda\,\frac{\partial f}{\partial\sigma_{ij}}\ . \qquad (2\text{-}6)$$

The quantity λ is a function of space and time defining the rate of energy dissipation at the respective point in space and time and is necessarily non-negative, $\lambda \geq 0$. In regions of plastic deformation inherently accompanied by energy dissipation, λ must be positive

$$\lambda > 0\ . \qquad (2\text{-}7)$$

For the v. Mises yield condition, Eq. (2-1),

$$\frac{\partial f}{\partial\sigma_{ij}} = s_{ij}$$

so that

$$\frac{1}{2}\,(\dot{u}_{i,j}+\dot{u}_{j,i}) = \frac{1}{2G}\,\dot{s}_{ij} + \frac{1-2\nu}{6G(1+\nu)}\,\dot{J}_1\delta_{ij} + \lambda s_{ij} \qquad (2\text{-}8)$$

where the strain rates are expressed in terms of the components $\dot{u}_i$ of the velocities

$$\dot{\epsilon}_{ij} = \frac{1}{2}\,(\dot{u}_{i,j} + \dot{u}_{j,i})\ . \qquad (2\text{-}9)$$

Finally, retaining only linear terms in the equations of motion, the latter become

$$\sigma_{ij,i} = \rho\,\frac{\partial\dot{u}_j}{\partial t}\ . \qquad (2\text{-}10)$$

The equations stated above, i.e., Eqs. (2-1), (2-8), and (2-10), apply in regions of plastic deformation accompanied by energy dissipation. In general, there will also be regions where at the particular time considered no plastic deformation or energy dissipation occurs. In such regions, while the strain rate-velocity relations (2-9) and the equations of motion (2-10) still hold, the yield condition (2-1) and the stress-strain relation (2-8) are no longer valid. Instead, in "non-dissipative" regions* the total strain rate is

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^E_{ij} \tag{2-11}$$

while

$$J_2' \leq k^2 . \tag{2-12}$$

The general equations are now applied to the following plane strain problem in Cartesian coordinates x, y, z. The originally stress-free half-space $x \geq 0$, Fig. 1, is subjected to time dependent, but uniform, loads on its surface $x = 0$,

$$\left.\begin{aligned} \sigma_x (0,y,z,t) &= \sigma_o(t) \\ \tau_{xy}(0,y,z,t) &= \tau_o(t) \\ \tau_{xz}(0,y,z,t) &\equiv 0 \end{aligned}\right\} \tag{2-13}$$

where the prescribed loads σ_o and τ_o vanish for $t < 0$. Because of the nature of the yield condition (2-1), the intensity of the shear is subjected to the restriction

$$| \tau_o(t) | \leq k . \tag{2-13a}$$

* "Non-dissipative" regions in the sense used here include neutral plastic regions where $J_2' = k^2$.

In a plane strain problem the stresses τ_{xz} and τ_{yz} and the displacements in the z-direction vanish, while all other quantities are independent of the coordinate z . In the present problem, the prescribed loads are also independent of y , so that the response too will be independent of y , and all quantities will be functions solely of x and t. When writing the various equations in scalar form, all derivatives with respect to y or z will therefore vanish.

In regions of plastic deformation and energy dissipation, $\lambda > 0$, Eq. (2-8) gives four meaningful scalar equations:

$$\frac{1}{2G}\dot{s}_x + \frac{1-2\nu}{6G(1+\nu)}\dot{J}_1 + \lambda s_x = \frac{\partial \dot{u}}{\partial x} \tag{2-14a}$$

$$\frac{1}{2G}\dot{s}_y + \frac{1-2\nu}{6G(1+\nu)}\dot{J}_1 + \lambda s_y = 0 \tag{2-14b}$$

$$\frac{1}{2G}\dot{s}_z + \frac{1-2\nu}{6G(1+\nu)}\dot{J}_1 + \lambda s_z = 0 \tag{2-14c}$$

$$\frac{1}{2G}\dot{\tau} + \lambda\tau = \frac{1}{2}\frac{\partial \dot{v}}{\partial x} \tag{2-14d}$$

where $\tau \equiv \tau_{xy}$ and $\dot{u}$, $\dot{v}$ are the x- and y-components of the velocity.

The equations of motion, Fq. (2-10), after the substitution $\sigma_x = s_x + \frac{1}{3}J_1$, give the two non-trivial relations

$$\left.\begin{aligned} \frac{\partial s_x}{\partial x} + \frac{1}{3}\frac{\partial J_1}{\partial x} &= \rho\frac{\partial \dot{u}}{\partial t} \\ \frac{\partial \tau}{\partial x} &= \rho\frac{\partial \dot{v}}{\partial t} \end{aligned}\right\} . \tag{2-15}$$

Subtracting Eq. (2-14c) from Eq. (2-14b), one finds

$$\frac{1}{2G}(\dot{s}_y-\dot{s}_z) + \lambda(s_y-s_z) = 0 \ .$$

Initially, at $t = 0$, s_y and s_z are both equal to zero, so that the above equation requires $s_y \equiv s_z$ for all values of x and t. Noting that by definition $s_x + s_y + s_z \equiv 0$, one finds

$$s_y = s_z = -\frac{1}{2}s_x \tag{2-16}$$

which is used to eliminate s_y and s_z.

Thus, the number of stress variables is reduced to three, namely, s_x, J_1, and τ. Simultaneously, the number of independent constituitive relations is also reduced to three. A convenient set, obtained by obvious manipulation of Eqs. (2-14) is

$$\left.\begin{aligned} \frac{1}{\beta G}\dot{J}_1 &= \frac{\partial\dot{u}}{\partial x} \\ \frac{1}{2G}\dot{s}_x + \lambda s_x &= \frac{2}{3}\frac{\partial\dot{u}}{\partial x} \\ \frac{1}{2G}\dot{\tau} + \lambda\tau &= \frac{1}{2}\frac{\partial\dot{v}}{\partial x} \end{aligned}\right\} \tag{2-17}$$

where

$$\beta \equiv \beta(\nu) \equiv \frac{2(1+\nu)}{1-2\nu} \ . \tag{2-18}$$

Using Eq. (2-16) leads finally to the yield condition in the simple form

$$\frac{3}{4}s_x^2 + \tau^2 = k^2 \ . \tag{2-19}$$

This relation, together with the five equations (2-15) and (2-17), are available to determine the six unknowns s_x , J_1, τ, $\dot{u}$, $\dot{v}$ and $\lambda > 0$ in regions of plastic energy dissipation which will be referred to as "dissipative" regions. There will also be complementary "non-dissipative" regions without plastic deformation or energy dissipation. In such regions the equality (2-1) is to be replaced by the inequality (2-12). As previously noted, the differential equations for the non-dissipative regions may be obtained by using $\epsilon_{ij}=\epsilon^E_{ij}$ instead of Eq. (2-8) in conjunction with Eqs. (2-9). This is equivalent to substituting $\lambda \equiv 0$ into Eqs. (2-14) giving

$$\left.\begin{aligned} \frac{1}{2G}\dot{s}_x + \frac{1-2\nu}{6G(1+\nu)}\dot{J}_1 &= \frac{\partial \dot{u}}{\partial x} \\ \frac{1}{2G}\dot{s}_y + \frac{1-2\nu}{6G(1+\nu)}\dot{J}_1 &= 0 \\ \frac{1}{2G}\dot{s}_z + \frac{1-2\nu}{6G(1+\nu)}\dot{J}_1 &= 0 \\ \frac{1}{2G}\dot{\tau} &= \frac{1}{2}\frac{\partial \dot{v}}{\partial x} \end{aligned}\right\} . \tag{2-20}$$

Noting that Eq. (2-16) is still valid in non-dissipative regions, and recalling the definition of β , Eqs. (2-20) may be reduced to

$$\left.\begin{aligned} \frac{1}{\beta G}\dot{J}_1 &= \frac{\partial \dot{u}}{\partial x} \\ \frac{1}{2G}\dot{s}_x &= \frac{2}{3}\frac{\partial \dot{u}}{\partial x} \\ \frac{1}{2G}\dot{\tau} &= \frac{1}{2}\frac{\partial \dot{v}}{\partial x} \end{aligned}\right\} . \tag{2-21}$$

Moreover, the inequality (2-12) becomes

$$\frac{3}{4} s_x^2 + \tau^2 \leq k^2 \; . \tag{2-22}$$

Subject to this inequality, the three equations (2-21) and the two equations of motion (2-15) form a system of five simultaneous differential equations for the five functions s_x , J_1 , τ , $\dot{u}$ and $\dot{v}$. This system, of course, is equivalent to the conventional equations governing elastic wave propagation.

The complete boundary value problem to be solved involves, therefore, an a priori unknown number of dissipative and non-dissipative regions with unknown and moving boundaries. The solution in each region is to be obtained from the appropriate set of equations given above, and from matching and external boundary conditions.

The solution of the boundary value problem formulated above for a general input $\sigma_o(t)$, $\tau_o(t)$ is a formidable problem, even if approached by numerical methods. As stated in the introduction, the basic set of non-linear differential equations is hyperbolic, and has two pairs of characteristic velocities, $\pm V_1$. and $\pm V_2$. The use of the standard numerical approach based upon the method of characteristics is, however, complicated by unknown moving boundaries between the regions and by the occurrence of discontinuities.

An alternative numerical approach is available which eliminates the difficulties caused by moving boundaries and by discontinuities. It replaces the differential equations by finite difference ones and uses artificial viscosity terms, as suggested in [4], in order to obtain some measure of approximation to the discontinuities which occur in the exact solution.

It should also be noted that for either of the two numerical approaches, a starting solution valid for a short time is required. The usual power series approach used to obtain a starting solution is unsuccessful if $\sigma_o(t)$ or $\tau_o(t)$ is discontinuous at $t = 0$, because the point $x = 0$, $t = 0$ then becomes a singularity.

The remainder of this paper* will not concern itself with the application of either of the mentioned numerical approaches to the general problem, but will consider only the much simpler situation when the applied surface loads are step functions in time. For this case solutions in closed form can be obtained. In addition to giving some idea of the nature of the response in the general case of time dependent loading, the results give a starting solution for a numerical treatment of the general problem.

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* Except for a discussion of open questions concerning the application of the method of characteristics in Appendix B.

III. HALF SPACE SUBJECTED TO COMBINED NORMAL AND TANGENTIAL STEP LOADS

Consider the case where the surface loads are step functions of time

$$\left.\begin{aligned} \sigma_o(t) &= \sigma_o h(t) \\ \tau_o(t) &= \tau_o h(t) \end{aligned}\right\} . \tag{3-1}$$

Using dimensional considerations, the number of dimensional quantities appearing in the equations of Sec. II may be reduced to four, i.e., x, t, G and ρ , by replacing the unknown variables s_x , J_1, τ, $\dot{u}$ and $\dot{v}$ by non-dimensional ones

$$\frac{s_x}{k} \; , \; \frac{J_1}{k} \; , \; \frac{\tau}{k} \; , \; \sqrt{\frac{\rho G}{k^2}} \; \dot{u} \; , \; \sqrt{\frac{\rho G}{k^2}} \; \dot{v} \tag{3-2}$$

respectively, while λ is expressed by a new function Λ , defined by

$$\lambda = \frac{1}{G} \dot{\Lambda} . \tag{3-3}$$

The four dimensional quantities may be arranged into a single non-dimensional independent variable

$$U = \sqrt{\frac{\rho}{G}} \; \frac{x}{t} \geq 0 . \tag{3-4}$$

All other possible non-dimensional combinations of x, t, G and ρ are functions of U , and are therefore equivalent to Eq. (3-4). Thus, the non-dimensional quantities (3-2) and (3-3) are solely functions of the independent non-dimensional variable, U, and of the non-dimensional parameters β, σ_o/k and τ_o/k.

A. DISSIPATIVE REGIONS

To convert the partial differential equations obtained in the previous section into ordinary differential equations with respect to U, note from Eq. (3-4)

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{1}{t}\sqrt{\frac{\rho}{G}}\,\frac{d}{dU} \qquad (3\text{-}5) \\ \frac{\partial}{\partial t} &= -\frac{U}{t}\,\frac{d}{dU} \qquad (3\text{-}6) \end{aligned}\right\} .$$

Using Eq. (3-3), the condition $\lambda > 0$ which must be satisfied in dissipative regions becomes

$$U\Lambda' < 0 \tag{3-7}$$

where $'$ denotes differentiation with respect to U.

The relations (2-17) assume the form

$$\left.\begin{aligned} \frac{U}{\beta}\,\frac{J_1'}{k} + \sqrt{\frac{\rho G}{k^2}}\,\dot{u}' &= 0 \\ \frac{U}{2}\,\frac{s_x'}{k} + \frac{s_x}{k}\,U\Lambda' + \frac{2}{3}\sqrt{\frac{\rho G}{k^2}}\,\dot{u}' &= 0 \\ \frac{U}{2}\,\frac{\tau'}{k} + \frac{\tau}{k}\,U\Lambda' + \frac{1}{2}\sqrt{\frac{\rho G}{k^2}}\,\dot{v}' &= 0 \end{aligned}\right\} \tag{3-8}$$

and the equations of motion, Eqs. (2-15), become

$$\left.\begin{aligned} \frac{s_x'}{k} + \frac{1}{3}\frac{J_1'}{k} + U\sqrt{\frac{\rho G}{k^2}}\;\dot{u}' &= 0 \\ \frac{\tau'}{k} + U\sqrt{\frac{\rho G}{k^2}}\;\dot{v}' &= 0 \end{aligned}\right\} . \tag{3-9}$$

The yield condition (2-19) in terms of s_x/k and τ/k is

$$\frac{3}{4}\left(\frac{s_x}{k}\right)^2 + \left(\frac{\tau}{k}\right)^2 = 1 \tag{3-10}$$

and its derivative with respect to U is

$$\frac{3}{4}\left(\frac{s_x}{k}\right)\frac{s_x'}{k} + \left(\frac{\tau}{k}\right)\frac{\tau'}{k} = 0 \ . \tag{3-11}$$

Equations (3-8), (3-9) and (3-11) form the following set of simultaneous, linear, homogeneous equations for the derivatives of the unknown functions:

$$\begin{bmatrix} 0 & \frac{U}{\beta} & 1 & 0 & 0 & 0 \\ \frac{U}{2} & 0 & \frac{2}{3} & 0 & 0 & \frac{s_x}{k} \\ 1 & \frac{1}{3} & U & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{U}{2} & \frac{1}{2} & \frac{\tau}{k} \\ 0 & 0 & 0 & 1 & U & 0 \\ \frac{3}{4}\frac{s_x}{k} & 0 & 0 & \frac{\tau}{k} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{s_x'}{k} \\ \frac{J_1'}{k} \\ \sqrt{\frac{\rho G}{k^2}}\;\dot{u}' \\ \frac{\tau'}{k} \\ \sqrt{\frac{\rho G}{k^2}}\;\dot{v}' \\ U\Lambda' \end{bmatrix} = 0 \ . \tag{3-12}$$

If the determinant of the system (3-12) does not vanish, $s'_x = J'_1 = \dot{u}' = \tau' = \dot{v}' = U\Lambda' \equiv 0$, and Eq. (3-7) being violated, the above equations inherently do not apply. On the other hand, solutions where $U\Lambda' < 0$ may exist if the determinant of Eqs. (3-12) vanishes, which gives the condition

$$\frac{3}{4} s_x^2 (1-U^2)(\beta-3U^2) - \tau^2 U^2 (4+\beta-3U^2) = 0 . \quad (3\text{-}13)$$

This condition, while necessary, is not sufficient to ensure $U\Lambda' < 0$, which remains to be proved later. Equation (3-13) expresses the variable U in terms of the stresses s_x and τ . However, since s_x and τ are related by the yield condition (3-10), either may be eliminated with the result

$$\frac{s_x}{k} = \pm \sqrt{\frac{4U^2(4+\beta-3U^2)}{3(\beta+U^2)}} \quad (3\text{-}14)$$

and

$$\frac{\tau}{k} = \pm \sqrt{\frac{(U^2-1)(3U^2-\beta)}{\beta + U^2}} . \quad (3\text{-}15)$$

The fact that U, s_x, and τ must be real results in restrictions on the extent of dissipative regions. The bounds on potential locations of dissipative regions depend on whether the value of β is greater than, less than, or equal to three:*

* For the usual range of Poisson's ratio, $0 \leq \nu < \frac{1}{2}$, the value β , Eq. (2-18), varies from 2 to ∞ .

For $\beta > 3$: $0 \leq U_1 < U_2 \leq 1$ or $\sqrt{\frac{\beta}{3}} \leq U_1 < U_2 \leq \sqrt{\frac{4+\beta}{3}}$ (3-16)

For $\beta < 3$: $0 \leq U_1 < U_2 \leq \sqrt{\frac{\beta}{3}}$ or $1 \leq U_1 < U_2 \leq \sqrt{\frac{4+\beta}{3}}$ (3-17)

For $\beta = 3$: $0 \leq U_1 < U_2 \leq \sqrt{\frac{7}{3}}$ (3-18)

where U_1 and U_2 designate the end points of a dissipative region. Plots of Eqs. (3-14) and (3-15) for typical values of β are shown in Figs. 2 and 3, respectively. As required by the two inequalities in each of Eqs. (3-16,17), there are two separate branches except for the special case $\beta = 3$ when the branches for s_x merge, while the ones for τ join in a cusp at $U = 1$.

It is interesting to note that the bounds $U = \sqrt{(4+\beta)/3}$ and $U = 1$, respectively, correspond to the velocities of P-waves and S-waves in an elastic medium. The bound $U = \sqrt{\beta/3}$ will be seen in Subsection B to correspond to the velocity of propagation of plastic shock fronts.

In a dissipative region where the determinant of Eqs. (3-12) vanishes, the variables J_1', $\sqrt{\rho G}\,\dot{u}'$, τ', $\sqrt{\rho G}\,\dot{v}'$ and $U\Lambda'$ can be written in terms of s_x'

$$J_1' = \frac{3\beta}{3U^2 - \beta}\, s_x' \qquad (3\text{-}19a)$$

$$\sqrt{\rho G}\,\dot{u}' = \frac{-3U}{3U^2 - \beta}\, s_x' \qquad (3\text{-}19b)$$

$$\tau' = -\frac{3}{4}\left(\frac{s_x}{\tau}\right) s_x' \qquad (3\text{-}19c)$$

$$\sqrt{\rho G}\,\dot{v}' = \frac{3}{4U}\left(\frac{s_x}{\tau}\right) s_x' \qquad (3\text{-}19d)$$

$$U\Lambda' = \frac{U(4+\beta-3U^2)}{2(3U^2-\beta)s_x}\, s_x' \ . \qquad (3\text{-}19e)$$

This set of equations becomes inapplicable for $U = \sqrt{\beta/3}$, where $\tau = 0$. The physical reason for the breakdown will become apparent in Subsection B.

It is now necessary to investigate whether Eq. (3-7), a prerequisite for dissipative regions, is satisfied everywhere in the ranges given by Eqs. (3-16 to 18). Using Eq. (3-14) to express s_x and s'_x in terms of U , Eq. (3-19e) becomes

$$U\Lambda' = \frac{3[U^2 + \beta + \frac{2}{\sqrt{3}}\sqrt{\beta(\beta+1)}][U^2 + \beta - \frac{2}{\sqrt{3}}\sqrt{\beta(\beta+1)}]}{2(\beta+U^2)(\beta-3U^2)} < 0 \tag{3-20}$$

or

$$\frac{U^2 + \beta - \frac{2}{\sqrt{3}}\sqrt{\beta(\beta+1)}}{\beta - 3U^2} < 0 . \tag{3-21}$$

Eq. (3-21) is valid if either

$$U^2 > \frac{2}{\sqrt{3}}\sqrt{\beta(\beta+1)} - \beta \quad \text{and} \quad U^2 > \frac{\beta}{3} \tag{3-22}$$

or

$$U^2 < \frac{2}{\sqrt{3}}\sqrt{\beta(\beta+1)} - \beta \quad \text{and} \quad U^2 < \frac{\beta}{3} . \tag{3-23}$$

In the pertinent range, $\beta \geq 2$, the following inequalities apply:

$$\text{For} \quad \beta > 3: \quad 1 < \frac{2}{\sqrt{3}}\sqrt{\beta(\beta+1)} - \beta < \frac{\beta}{3} \tag{3-24}$$

$$\text{For} \quad \beta < 3: \quad \frac{\beta}{3} < \frac{2}{\sqrt{3}}\sqrt{\beta(\beta+1)} - \beta < 1 . \tag{3-25}$$

Thus, for $\beta \neq 3$ Eq. (3-22) is satisfied in one range of each of Eqs. (3-16) and (3-17), while Eq. (3-23) is satisfied in the other ranges. The regions described by either of Eqs. (3-16) or (3-17) are consequently possible regions of plastic dissipation. For the special case $\beta = 3$, Eq. (3-21) is satisfied in the range, Eq. (3-18). All branches of the curves of s_x and τ against U, shown in Figs. 2 and 3, represent therefore valid solutions.

Equations (3-14) and (3-15) already give s_x and τ as functions of U, while the remaining quantities J_1, $\dot{u}$ and $\dot{v}$ can be obtained by integration from Eqs. (3-19a,b,d), respectively. To determine J_1, Eq. (3-19a) may be integrated by parts

$$\frac{J_1(U)}{k} = \frac{3\beta}{3U^2-\beta}\left(\frac{s_x}{k}\right) + \int \frac{6U\left(\frac{s_x}{k}\right)dU}{(3U^2-\beta)^2} + \text{const.} \qquad (3\text{-}26)$$

After a series of manipulations shown in detail in Appendix A, the integral in Eq. (3-26) is obtained in closed form in terms of elliptic integrals,

$$\frac{J_1(U)}{k} - \frac{J_1(U_2)}{k} = H(U) - H(U_2) \qquad (3\text{-}27)$$

where $J_1(U_2)$ is the value of J_1 at the boundary U_2 of the region. When $0 < U < \sqrt{\beta/3}$ the function $H(U)$ is

$$\mp H(U) = \frac{3\sqrt{3}}{\sqrt{\beta+1}} F(\phi,\bar{k}) + \sqrt{3(\beta+1)}\, E(\phi,\bar{k})$$
$$+ \frac{3\sqrt{3}}{4}\frac{(\beta-3)}{\sqrt{\beta+1}} \Pi\left(\phi,\frac{1}{4},\bar{k}\right) - \frac{\sqrt{3}}{2}(\beta-3)\tanh^{-1}\left(\frac{3U}{\beta}\sqrt{\frac{\beta+U^2}{4+\beta-3U^2}}\right) \qquad (3\text{-}28)$$

where the sign of $H(U)$ must be chosen to correspond with that of s_x in Eq. (3-14). F, E, and Π are the elliptic integrals of the first, second, and third kind, respectively, and

$$\sin \phi = \sqrt{\frac{4U^2(\beta+1)}{(\beta+4)(\beta+U^2)}} \tag{3-29}$$

$$\bar{k}^2 = \frac{\beta + 4}{4(\beta+1)} \; . \tag{3-30}$$

When $\sqrt{\beta/3} < U < \sqrt{(4+\beta)/3}$ the term $\tanh^{-1}$ in Eq. (3-28) is to be replaced by $\coth^{-1}$. The function $H(U)$ is well-behaved except for $U = \sqrt{\beta/3}$ where the $\tanh^{-1}$ (or $\coth^{-1}$) term becomes infinite. When $\beta = 3$, the term does not appear.

Similar integrals for the velocities $\dot{u}$ and $\dot{v}$ are also evaluated in Appendix A. The expression for $\dot{u}$ involves only transcendental functions, while the one for $\dot{v}$ is in terms of elliptic integrals, except for $\beta = 3$, where the elliptic integrals reduce to simpler transcendental functions.

B. DISSIPATIVE SHOCK FRONTS

The wave propagation problem under study being hyperbolic, the possibility of discontinuities in the solutions must also be considered, and in addition to dissipative regions, potential locations of fronts of discontinuity involving energy dissipation must be explored. The general study, [5], of elastic-plastic wave propagation shows that, in an ideal elastic-plastic material subject to the yield condition (2-1), fronts of discontinuity can occur only in locations where the shear stress tangential to the front vanishes. The discontinuities are restricted to the mean stress, $J_1/3$, and to the particle velocity normal to the front, the velocity of which is $\sqrt{K/\rho}$, where $K = \beta G/3$ is the (elastic) bulk modulus. The complete solution of the problem considered here being a function of U as defined by Eq. (3-4), such a front, if it occurs, is located at

$$U = \sqrt{\frac{\beta}{3}} \quad . \tag{3-31}$$

The requirement $\tau = 0$ defines the value of s_x at the front

$$\frac{s_x}{k} = \pm \frac{2}{\sqrt{3}} \tag{3-32}$$

while the discontinuities ΔJ_1 and $\Delta \dot{u}$ are related by momentum considerations

$$\Delta J_1 + \sqrt{3\beta\rho G}\, \Delta \dot{u} = 0 \ . \tag{3-33}$$

In addition, the condition that energy at the front is dissipated requires

$$\Delta J_1 / s_x > 0 \tag{3-34}$$

where "Δ" signifies the value behind the front, minus the value ahead.

The fact that a discontinuity in the solutions may occur at $U = \sqrt{\beta/3}$ is the cause of the breakdown of the differential relations (3-19) noted previously.

C. NON-DISSIPATIVE REGIONS AND NON-DISSIPATIVE SHOCK FRONTS

In non-dissipative regions, the governing equations (2-15) and (2-21), when written in terms of the non-dimensional variables (3-2), and of the independent variable U become

$$
\begin{bmatrix}
0 & \frac{U}{\beta} & 1 & 0 & 0 \\
\frac{U}{2} & 0 & \frac{2}{3} & 0 & 0 \\
1 & \frac{1}{3} & U & 0 & 0 \\
0 & 0 & 0 & \frac{U}{2} & \frac{1}{2} \\
0 & 0 & 0 & 1 & U
\end{bmatrix}
\begin{bmatrix}
\frac{s_x'}{k} \\
\frac{J_1'}{k} \\
\sqrt{\frac{\rho G}{k^2}}\ \dot{u}' \\
\frac{\tau'}{k} \\
\sqrt{\frac{\rho G}{k^2}}\ \dot{v}'
\end{bmatrix}
= 0 \qquad (3\text{-}35)
$$

while the yield condition Eq. (2-22) becomes

$$
\frac{3}{4}\left(\frac{s_x}{k}\right)^2 + \left(\frac{\tau}{k}\right)^2 \leq 1 \ . \qquad (3\text{-}36)
$$

Relations (3-35) being a set of homogeneous linear equations, it follows that

$$
s_x' = J_1' = \dot{u}' = \tau' = \dot{v}' = 0 \qquad (3\text{-}37)
$$

unless the determinant of the system vanishes, i.e.,

$$
U(1-U^2)(4+\beta-3U^2) = 0 \ . \qquad (3\text{-}38)
$$

Separating each non-dissipative region which contains positive roots* of Eq. (3-38) into two or three regions excluding these roots, Eqs. (3-37) require that all stresses and velocities within the regions formed in this manner are constant.

---

* The negative values of U are of no consequence because $U \geq 0$, and the root $U = 0$ can be shown to be trivial.

Without further analysis, one expects in a non-dissipative region the discontinuities known from the theory of plane elastic waves, i.e., P- and S-fronts, and indeed the positive roots of Eq. (3-38)

$$U_P = \sqrt{\frac{4+\beta}{3}} \quad , \quad U_S = 1 \tag{3-39}$$

define locations moving with the proper velocities. The solution may therefore have discontinuities at these points. At U_P discontinuities $\Delta\sigma_x$, Δs_x , ΔJ_1 , and $\Delta\dot{u}$ may occur, related by

$$\Delta\sigma_x : \Delta s_x : \Delta J_1 : \sqrt{\rho G}\,\Delta\dot{u} = 1 : \frac{4}{4+\beta} : \frac{3\beta}{4+\beta} : -\sqrt{\frac{3}{4+\beta}} \tag{3-40}$$

while at U_S discontinuities $\Delta\tau$ and $\Delta\dot{v}$ are possible, where

$$\Delta\tau = -\sqrt{\rho G}\,\Delta\dot{v} \ . \tag{3-41}$$

Obviously, the intensity of the discontinuities must not violate the yield condition (3-36).

IV. CONSTRUCTION OF SOLUTIONS

In the previous section, solutions of the basic differential equations have been obtained for regions with and without energy dissipation, not knowing the extent of these regions in the complete solution to be found. Further, there are three potential locations of discontinuities, one with, and two without dissipation. These partial solutions must now be combined to form a complete solution satisfying the externally prescribed conditions. Due to the multitude of possible combinations, and due to the lack of a general existence and uniqueness theorem, it is imperative to use a systematic procedure in the construction of solutions to be certain that no solution is overlooked.

The yield condition, Eq. (2-1), being an even function of the components of stress, the complete solutions will also be even in σ_o and in τ_o. Without loss of generality, the following will, therefore, consider only the case of compressive surface pressure, $\sigma_o < 0$, and positive shear, $0 \leq \tau_o \leq k$. Consequently, the lower sign must be taken for s_x in Eq. (3-14), and correspondingly, in Eq. (3-28) and in Appendix A for the functions $H(U)$, $H_u(U)$ and $H_v(U)$.

The prescribed surface loading, Eq. (3-1), leads to the boundary conditions for $U = 0$,

$$\left.\begin{aligned} \sigma_x(0) = s_x(0) + \frac{1}{3} J_1(0) = \sigma_o \\ \tau(0) = \tau_o \end{aligned}\right\} \qquad (4\text{-}1)$$

where the nature of the material requires $|\tau_o| \leq k$. The half-space being originally stress free and at rest, all stress components and velocities vanish for $U \to \infty$, furnishing the boundary conditions

$$\lim_{U \to \infty} [s_x, J_1, \tau, \epsilon_x, \epsilon_y, \dot{u}, \dot{v}] \equiv 0 \ . \tag{4-2}$$

Because the solutions in dissipative regions change in character depending upon the value of β, separate treatment for the cases $\beta \geq 3$ and $\beta < 3$ is required.

A. SOLUTIONS FOR $\beta \geq 3$

Starting with the fact that for $U \to \infty$ all stresses vanish, the solutions will be developed by consideration of the conditions for gradually decreasing values of U. The stresses given by Eq. (4-2) do not satisfy the yield condition (3-10), and the first changes, from vanishing stresses to initiation of yield, must therefore be elastic. Such changes can only be discontinuities at $U = U_p \equiv \sqrt{(4+\beta)/3}$ and at $U = U_s \equiv 1$, described by Eqs. (3-40) and (3-41).

Case A.1 a combination of discontinuous changes $\Delta\sigma_x$ and $\Delta\tau$ in their respective locations are selected so that yield according to Eq. (3-10) is not reached, no dissipative regions or shock can occur anywhere, and the stresses introduced at the P- and S-fronts must remain constant up to the surface, $U = 0$. The resulting, completely elastic solution leads to surface loads

$$\left.\begin{aligned} \sigma_o &= \Delta\sigma_x \\ \tau_o &= \Delta\tau \end{aligned}\right\} \ . \tag{4-3}$$

If the combination of $\Delta\sigma_x$ and $\Delta\tau$ just satisfies the yield condition, but no dissipative region is utilized, the above solution still applies. It is therefore valid for the range

$$0 > \frac{\sigma_o}{k} \geq - \frac{4 + \beta}{2\sqrt{3}} \sqrt{1 - \left(\frac{\tau_o}{k}\right)^2} \equiv \frac{\sigma_1}{k} . \qquad (4\text{-}4)$$

Case A.2 Consider next the situation when the combination of $\Delta\sigma_x$ and $\Delta\tau$ satisfies the yield condition (3-10), and $\Delta\tau \neq 0$. For $\beta > 3$ the yield condition being satisfied only for values $U \leq 1$, the plastic shock at $U = \sqrt{\beta/3} > 1$ cannot occur, and of the two potential ranges for dissipative regions, Eq. (3-16), only one

$$0 \leq U_1 < U_2 \leq 1$$

remains to be considered. For $\beta = 3$, the value of the shear $\tau = \Delta\tau \neq 0$ at $U = \sqrt{\beta/3} \equiv 1$ also precludes a plastic shock, and the above range is again the only one to be considered. At the terminal point U_2 of the dissipative region, if any, the value of the shear stress must be equal to that in the adjoining non-dissipative region, $\tau = \Delta\tau$. Figure 3 shows τ as a function of U for typical values of β . For any β , there is one, and just one, value U_2 which corresponds to $\tau = \Delta\tau$, where a region of plastic dissipation may begin. The other end point, $U_1 < U_2$, of the dissipative region, may be selected at will, so that it may correspond to any value of $\tau_o > \Delta\tau$. As U_1 approaches U_2 , the value τ_o approaches $\Delta\tau$, until in the limit the solution for $\sigma_o = \sigma_1$ obtained in A.1 is again reached. The region $U_1 \geq U \geq 0$ is necessarily non-dissipative with constant stresses.

The entire stress history is obtained by computing s_x from Eq. (3-14), τ from (3-15), and J_1 from (3-27) and (3-28). The normal stress at the surface, $\sigma_o = s_x(U_1) + \frac{1}{3}J_1(U_1)$, is a function of τ_o, $\Delta\tau$ and of the corresponding values U_1 and U_2,

$$\frac{\sigma_o}{k} = -\frac{2}{\sqrt{3}}\sqrt{1-\left(\frac{\tau_o}{k}\right)^2} - \frac{\beta}{2\sqrt{3}}\sqrt{1-\left(\frac{\Delta\tau}{k}\right)^2} + \frac{1}{3}H(U_1) - \frac{1}{3}H(U_2) . \tag{4-5}$$

This expression for σ_o depends on $\Delta\tau$ through the second and fourth terms only, both contributions having the same sign. Both terms vary monotonically, and because U_2 increases with decreasing $\Delta\tau$, $\sqrt{1-(\Delta\tau/k)^2}$ and $H(U_2)$ both vary inversely to $\Delta\tau$. For a given value of τ_o, the largest surface pressure, σ_2, is therefore obtained for $\Delta\tau \to 0$. Thus, Case A.2 applies if

$$-\sigma_1 \leq -\sigma_o \leq -\sigma_2 \tag{4-6}$$

where σ_1 is defined by Eq. (4-4), while σ_2 is an implicit function of τ_o and β,

$$\frac{\sigma_2}{k} \equiv -\frac{2}{\sqrt{3}}\sqrt{1-\left(\frac{\tau_o}{k}\right)^2} - \frac{\beta}{2\sqrt{3}} + \frac{1}{3}H(U_1[\tau_o,\beta],\beta) - \frac{1}{3}H(1,\beta) . \tag{4-7}$$

The velocity history may be obtained from Eqs. (3-40) and (3-41), and from Appendix A.

<u>Case A.3</u> Finally, let the discontinuity $\Delta\sigma_x$ be such that the yield condition will be satisfied by the discontinuity $\Delta\sigma_x$ at $U = U_P$ alone,

$$\frac{\Delta\sigma_x}{k} = -\frac{4+\beta}{2\sqrt{3}} . \tag{4-8}$$

The yield condition being satisfied at $U \leq U_P$, both of the two dissipative regions, Eq. (3-16), must be considered. However, the shear τ just behind the P-front being zero, Fig. 3 shows that no dissipative region can actually occur for $U \geq \sqrt{\beta/3}$ because the starting and end points of such a region would be identical, $U_1 = U_2 = \sqrt{\beta/3}$. The stresses in the range $U_P > U > \sqrt{\beta/3}$ can therefore not vary and are given by Eq. (4-8). At $U = \sqrt{\beta/3}$, the necessary condition $\tau = 0$ being satisfied, a dissipative shock may occur. The strength of this shock, described by ΔJ_1, is arbitrary except for the sign,

$$\Delta J_1 \leq 0 . \tag{4-9}$$

In the range $\sqrt{\beta/3} > U > 1$, there is no alternative to a non-dissipative region of constant stress. At $U = 1$, the yield condition excludes the possibility of an elastic change in shear, but Fig. 3 indicates that a dissipative region is possible from $U_2 = 1$, corresponding to $\tau = 0$, to any desired value U_1, corresponding to the surface shear τ_0. The surface pressure is given by

$$\sigma_0 = \sigma_2 + \frac{1}{3} \Delta J_1 . \tag{4-10}$$

If ΔJ_1 is selected equal to zero, the solution obtained is identical to that obtained in Case A.2 for $\sigma_0 = \sigma_2$. By using sufficiently large values of $|\Delta J_1|$, one obtains solutions for any value $|\sigma_0| > |\sigma_2|$ of the surface pressure without upper bound. Equation (4-10) defining ΔJ_1 uniquely as a function of $|\sigma_0| \geq |\sigma_2|$, the solutions in the range are again unique, and do not overlap Case A.2.

The corresponding velocity history may be obtained from Eq. (3-33) in addition to the relations noted in A.2.

The three cases considered exhaust all the possible combinations of regions and fronts for $\beta \geq 3$. There is one and only one solution for any possible combination of surface loads σ_o and τ_o. The regions in the σ_o, τ_o-plane governing each case and the separation lines σ_1 and σ_2 are shown in Fig. 4, drawn for $\beta = 5$. The appropriate expressions for the solution are listed in Tables I to III. Typical distributions of σ_x and τ for the non-elastic cases are shown in Figs. 5 and 6.

B. SOLUTIONS FOR $\beta < 3$

Reasoning exactly as for $\beta \geq 3$, the stresses must vanish for large values of U until the P- or S-fronts are reached, and elastic discontinuities in stress, $\Delta\sigma_x$ and $\Delta\tau$, occur to initiate yield.

Case B.1 If the discontinuities $\Delta\sigma_x$ and $\Delta\tau$ are sufficiently small so that yield is not reached, the situation is exactly as in Case A.1 for $\beta \geq 3$ and Eqs. (4-3) and (4-4) apply.

Case B.2 If the stress discontinuities $\Delta\sigma_x$ and $\Delta\tau$ are such that the yield condition is just satisfied, but $\Delta\tau \neq 0$, a dissipative region or shock is impossible unless $U < 1$. Due to $\Delta\tau \neq 0$, the dissipative shock at $U = \sqrt{\beta/3} < 1$ cannot occur, since it requires $\tau = 0$, but a dissipative region may start at a point $U_2 < \sqrt{\beta/3}$ corresponding to $\tau = \Delta\tau$. Continuing this region to a point $U_1 < U_2$ corresponding to $\tau = \tau_o > \Delta\tau$, a solution is obtained for a value σ_o, not known a priori. As in Case A.2, the entire stress history is obtained from Eqs. (3-14), (3-15), (3-27), and (3-28), while the normal stress at the surface, σ_o, is given by Eq. (4-5). The dependence of σ_o on $\Delta\tau$ is again monotonic so that the solutions found in the range are unique. It is important to note that the function $H(U)$ becomes infinite as U approaches $\sqrt{\beta/3}$. Therefore, for a given τ_o, it is evident that in the limit $\Delta\tau \rightarrow 0$, $U_2 \rightarrow \sqrt{\beta/3}$, and the surface pressure will increase without bound, $\sigma_o \rightarrow -\infty$.

The solution described applies, therefore, for any value of the surface pressure not covered by Case B.1, except that the condition $\tau_o > \Delta\tau \neq 0$ excludes the case $\tau_o = 0$.

Case B.3 The remaining possibility is a discontinuity $\Delta\sigma_x$ at U_p, sufficient to satisfy the yield condition. In the adjoining non-dissipative region, $\sigma_x = \Delta\sigma_x$ and $\tau=\Delta\tau=0$, which permits a dissipative shock of arbitrary strength $\Delta J_1 \leq 0$ at $U = \sqrt{\beta/3}$. This shock may be followed by a non-dissipative region of constant stress, furnishing solutions for $\tau_o = 0$ for any value of σ_o not covered by Case B.1.

If one attempts, starting as in the previous paragraph, to employ a dissipative region behind the shock, i.e., from $U_2 = \sqrt{\beta/3}$ to some value $U_1 < U_2$, the singularity in $H(U)$ at $\sqrt{\beta/3}$ leads to $\sigma_o = -\infty$, a physically unrealistic situation.

The three cases considered furnish, therefore, one and only one solution for each combination of σ_o and τ_o. The regions in the σ_o, τ_o-plane governing each case are shown in Fig. 7. The appropriate expressions for the solution are listed in Tables I, II, and IV. Typical distributions of σ_x and τ for Case B.2 are shown in Fig. 8. The velocity history for $\beta < 3$ may be obtained in a manner analogous to that for $\beta \geq 3$.

V. SUMMARY OF RESULTS AND DISCUSSION

The stresses and velocities in an ideal elastic-plastic half-space, subject to the v. Mises yield condition, have been obtained due to normal and shear stresses on the surface, both of which are step functions of the time. The solutions are in closed form, containing elliptic integrals and simpler transcendental functions. Expressions for the stresses, including pertinent basic relations, are tabulated in Tables I to IV, while the expressions for the velocities are given in Appendix A. The results indicate a number of ranges where different solutions apply, depending on the intensity of the surface loads, and on the value of a material parameter $\beta = \frac{2(1+\nu)}{1 - 2\nu}$ which varies with Poisson's ratio. The dependence of these ranges upon the applied surface stresses σ_o and τ_o is shown in Fig. 4 for $\beta \geq 3$, and in Fig. 7 for $\beta < 3$.

Excluding the cases where σ_o and τ_o are so small that no plastic deformation occurs, Figs. 5, 6, and 8 show the stresses for a typical case of each of the major situations. Figures 5 and 6 apply if $\beta \geq 3$, which corresponds to Poisson's ratio $\nu \geq 1/8$. The non-dimensional ordinate U in the plots is proportional to x/t so that the figures illustrate the x-distribution of the stresses at any time t. For an applied compressive stress, $\sigma_o < 0$, in the range $-\sigma_1(\tau_o,\beta) < -\sigma_o < -\sigma_2(\tau_o,\beta)$, the stress plot is given by Fig. 5. There is an elastic precursor consisting of a P-wave of uniform strength, followed by an S-wave, also of uniform strength, so that the combination of the two states of stresses just satisfies the yield condition. Proceeding towards the surface there is no plastic deformation or energy dissipation until a point $U = U_2$ is reached. Plastic deformation accompanied by energy dissipation occurs in a region from U_2 to U_1. In this region, σ_x and τ increase until the surface values σ_o and τ_o are reached. Finally, between the end of the plastic region at U_1, and the surface, the stresses are constant. If $-\sigma_o > -\sigma_2$, Fig. 6, there is again an elastic precursor consisting of a P-wave of uniform strength so that the yield condition is just satisfied. At $U = \sqrt{\beta/3}$ there is a plastic shock, a discontinuity in σ_x without change in τ, followed by a region of constant stress. From $U = 1$ to $U = U_1$, there is a plastic region where σ_x and τ increase, just as in Fig. 5, until the surface values σ_o, τ_o are reached. For $0 < U < U_1$, there is again a region of constant stress.

For $\beta < 3$, $\nu < 1/8$, Fig. 8 applies for $-\sigma_o > -\sigma_1(\tau_o, \beta)$ and $\tau_o \neq 0$. The stress history in Fig. 8 is similar to Fig. 5 for $\beta \geq 3$, but there is an important difference. As the end point U_2 of the plastic region approaches $\sqrt{\beta/3}$, the surface pressure $-\sigma_o$ increases beyond any bound, so that a solution with a plastic shock never occurs, except if $\tau_o = 0$.

In view of the fact that no general existence and uniqueness theorem for the problem of wave propagation in elastic-plastic media is available, it is pertinent to note that solutions were found, for any combination of Poisson's ratio ν, surface pressure $-\sigma_o$, and surface shear $|\tau_o| \leq k$, and that these solutions are unique.

While the method of solution used in Sec. III does not appear to utilize the characteristics of the system of partial differential equations, the characteristic velocities appear in the analysis in a disguised form. Inverting Eqs. (3-14, 15), so that the non-dimensional variable U, Eq. (3-4), becomes a double valued function of the state of stress, it is shown in Appendix B, that the characteristic velocities V_i are related to the values of U,

$$V_i = \pm \sqrt{\frac{G}{\rho}}\, U .$$

Figures 2 and 3 demonstrate, therefore, the dependence of the characteristic velocities V_i on the values of s_x or τ.

The availability of the characteristic velocities V_i, and of the relations along the characteristics, also given in Appendix B, are an invitation to their use for numerical analysis in the case of general surface stresses $\sigma_o(t)$ and $\tau_o(t)$. However, the moving boundaries between dissipative and non-dissipative regions pose difficulties requiring further study as outlined in Appendix B.

The approach used in Sec. III is readily applicable to other types of yield conditions. The case for the Tresca condition becomes slightly more involved than for the v. Mises condition, since there are several sub-cases depending on whether or not σ_z is one of the extreme principal stresses entering the yield condition, $|\sigma_{max} - \sigma_{min}| = 2k$. In the sub-case which controls in the present problem, σ_z is the intermediate principal stress and the relations between s_x or τ and U, Eqs. (3-14, 15), change, but without affecting the nature of the solutions and with only minute numerical effects.

Another potential generalization of the present paper is a two-dimensional problem, the determination of the steady-state solution for an elastic-plastic half-space subjected to a step pressure on the surface $x = 0$, moving with super-seismic velocity V_L. Using a system of polar coordinates moving with the front of the load, Fig. 9, dimensional considerations require the stresses to be functions of the angle θ, but not of the radius r. This permits the reduction of the partial differential equations into a set of ordinary ones, containing, however, a larger number of unknown dependent variables than in the problem treated here.

APPENDIX A

INTEGRATIONS REQUIRED FOR THE DETERMINATION OF J_1, $\dot{u}$, and $\dot{v}$

1. EVALUATION OF J_1

Starting with Eq. (3-26) in the text, substitution of the value of s_x/k from Eq. (3-14) yields

$$\pm \frac{J_1(U)}{k} = \frac{2U\beta\sqrt{3}}{3U^2 - \beta} \sqrt{\frac{4+\beta-3U^2}{\beta + U^2}}$$

$$+ 6\beta\sqrt{3} \int \sqrt{\frac{U^2(4+\beta-3U^2)}{\beta + U^2}} \frac{2UdU}{(3U^2 - \beta)^2} \pm \text{const.} \tag{A-1}$$

requiring the evaluation of the integral

$$I(U) \equiv \int_0^U \sqrt{\frac{\xi^2(4+\beta-3\xi^2)}{\beta + \xi^2}} \frac{2\xi d\xi}{(3\xi^2-\beta)^2} = \int_0^{U^2} \sqrt{\frac{\eta(4+\beta-3\eta)}{\beta + \eta}} \frac{d\eta}{(3\eta-\beta)^2} . \tag{A-2}$$

Multiplying numerator and denominator in the last integral by

$$\sqrt{\eta(4+\beta-3\eta)}$$

$I(U)$ becomes

$$I(U) = \int_0^{U^2} \frac{\dfrac{\eta(\frac{4+\beta}{3} - \eta)}{3\sqrt{3}\,(\eta - \frac{\beta}{3})^2}\, d\eta}{\sqrt{(\frac{4+\beta}{3} - \eta)\eta(\eta+\beta)}} \tag{A-3}$$

which is of the form

$$\int_b^y \frac{R(t)dt}{\sqrt{(a-t)(t-b)(t-c)}}$$

where $R(t)$ is a rational function of t. This integral may be converted, Byrd and Friedman [6], No. 235.20, with $a \geq y > b > c$, to

$$\int_b^y \frac{R(t)dt}{\sqrt{(a-t)(t-b)(t-c)}} = g \int_0^{u_1} R\left(\frac{b-c\bar{k}^2 sn^2 u}{dn^2 u}\right) du \qquad \text{(A-4)}$$

where*

$$\left.\begin{aligned} sn^2 u &= \frac{(a-c)(t-b)}{(a-b)(t-c)} \\ \bar{k}^2 &= \frac{a-b}{a-c} \\ g &= \frac{2}{\sqrt{a-c}} \\ \phi &= \text{am } u_1 = \sin^{-1}\sqrt{\frac{a-c)(y-b)}{(a-b)(y-c)}} \\ snu_1 &= \sin\phi \end{aligned}\right\} \qquad \text{(A-5)}$$

and

$$a = \frac{4+\beta}{3}\ , \quad b = 0\ , \quad c = -\beta \quad \text{and} \quad y = U^2 \qquad \text{(A-6)}$$

while

$$R(\eta) = \frac{\eta(\frac{4+\beta}{3} - \eta)}{3\sqrt{3}\ (\eta - \frac{\beta}{3})^2}\ .$$

---

* Note the difference in meaning of the symbols U and u.

Substituting these values into Eq. (A-5), one has

$$\left.\begin{aligned} sn^2u &= \frac{4(\beta+1)\eta}{(\beta+4)(\beta+\eta)} \\ \bar{k}^2 &= \frac{\beta+4}{4(\beta+1)} \\ g &= \sqrt{\frac{3}{\beta+1}} \\ \sin\phi = snu_1 &= \sqrt{\frac{4(\beta+1)U^2}{(\beta+4)\ (\beta+U^2)}} \end{aligned}\right\} \quad . \qquad \text{(A-7)}$$

The argument of R which appears on the right side of Eq. (A-4) is

$$\frac{\beta(\beta+4)}{4(\beta+1)}\ \frac{sn^2u}{dn^2u}$$

and the integral (A-3) finally becomes

$$I(U) = \frac{(\beta+4)^2}{\beta\sqrt{\beta+1}} \int_0^{u_1} \frac{sn^2u[4(\beta+1)dn^2u - 3\beta\ sn^2u]}{[4(\beta+1)dn^2u - 3(\beta+4)sn^2u]^2}\, du \quad . \qquad \text{(A-8)}$$

Noting the relations (Pierce [7], Nos. 730, 731)

$$sn^2u + cn^2u = 1$$

$$dn^2u + \bar{k}^2 sn^2u = 1$$

Eq. (A-8) reduces further to

$$I(U) = \frac{(\beta+4)^2}{4\beta(\beta+1)^{\frac{3}{2}}} \int_0^{u_1} \frac{sn^2u\ cn^2u}{(1 - \frac{\beta+4}{\beta+1} sn^2u)^2}\, du \quad . \qquad \text{(A-9)}$$

Applying [6] No. 362.18 for $\alpha^2 = \frac{\beta+4}{\beta+1}$, $\bar{k}^2 = \frac{3+4}{4(\beta+1)} = \frac{\alpha^2}{4}$ gives

$$I(U) = \frac{\sqrt{\beta+1}}{6\beta}\left[\frac{3}{4}\left(\frac{\beta-3}{\beta+1}\right)\Pi(\phi,\alpha^2,\bar{k}) - E(\phi,\bar{k}) - \frac{3}{4}F(\phi,\bar{k})\right.$$

$$\left. + \left(\frac{\beta+4}{\beta+1}\right)\frac{\mathrm{sn}u_1\,\mathrm{cn}u_1\,\mathrm{dn}u_1}{1-\frac{\beta+4}{\beta+1}\mathrm{sn}^2u_1}\right] \qquad \text{(A-10)}$$

where

$$u \equiv F(\phi,\bar{k}) \quad \equiv \text{elliptic integral of the first kind}$$

$$E(u) \equiv E(\phi,\bar{k}) \quad \equiv \text{elliptic integral of the second kind}$$

$$\Pi(u,\alpha^2) \equiv \Pi(\phi,\alpha^2,\bar{k}) \equiv \text{elliptic integral of the third kind}$$

and

$$\left.\begin{aligned} \mathrm{sn}u_1 &= \sin\phi = \sqrt{\frac{4(\beta+1)\ U^2}{(\beta+4)(\beta+U^2)}} \\ \mathrm{cn}u_1 &= \cos\phi = \sqrt{\frac{\beta(4+\beta-3U^2)}{(\beta+4)(\beta+U^2)}} \\ \mathrm{dn}u_1 &= \sqrt{1-\bar{k}^2\sin^2\phi} = \sqrt{\frac{\beta}{\beta+U^2}} \end{aligned}\right\} \ . \qquad \text{(A-11)}$$

Using Eq. (A-11), the last term in Eq. (A-10) becomes

$$\left(\frac{\beta+4}{\beta+1}\right)\frac{\mathrm{sn}u_1\,\mathrm{cn}u_1\,\mathrm{dn}u_1}{1-\left(\frac{\beta+4}{\beta+1}\right)\mathrm{sn}^2u_1} = \frac{2\beta U}{\beta-3U^2}\sqrt{\frac{4+\beta-3U^2}{(\beta+1)(\beta+U^2)}} \ . \qquad \text{(A-12)}$$

Returning to the original expression (A-1), noting that $I(0) = 0$ and that expression (A-12) cancels with the first term on the right side of Eq. (A-1), one has

$$\frac{J_1(U)}{k} = H(U) + \text{const.} \tag{A-13}$$

where

$$H(U) = \mp \sqrt{3(\beta+1)}\left[\frac{3}{4} F(\phi,\bar{k}) + E(\phi,\bar{k}) - \frac{3}{4}\left(\frac{\beta-3}{\beta+1}\right)\Pi(\phi,\alpha^2,\bar{k})\right]. \tag{A-14}$$

If the constant of integration in Eq. (A-13) is expressed by the value of J_1/k at the end of the region U_2 , then

$$\frac{J_1(U)}{k} - \frac{J_1(U_2)}{k} = H(U) - H(U_2) \ . \tag{A-15}$$

For any given value of U , in a dissipative region, ϕ may be found from relation (A-11) while the F and E functions may be taken from any of several tables of elliptic integrals, e.g., [8]. The integral of the third kind, Π , is tabulated in Selfridge and Maxfield [9] for various values of $\bar{k}$ but only for $\alpha^2 < 1$. "Additional formula" on page xi relates the tabulated values with those for $\alpha^2 > 1$ required here,

$$\Pi(\phi,\alpha^2,\bar{k}) + \Pi(\phi,\frac{\bar{k}^2}{\alpha^2},\bar{k}) = F(\phi,\bar{k})$$

$$+ \sqrt{\frac{\alpha^2}{(1-\alpha^2)\ (\alpha^2-\bar{k}^2)}}\ \tan^{-1}\left[\tan\phi\sqrt{\frac{(1-\alpha^2)(\alpha^2-\bar{k}^2)}{\alpha^2(1-\bar{k}^2\ \sin^2\phi)}}\right]. \tag{A-16}$$

Since $\alpha^2 > 1$ and $\alpha^2 > \bar{k}^2$, each of the radicals above will be imaginary. Equations 6 and 7 on page 51 in [10] may be used to obtain the last term of Eq. (A-16) as a real function of a real argument.

Let the value sought be

$$\psi = \frac{1}{i} \tan^{-1} i\chi \qquad \text{(A-17)}$$

where χ is real and either $\gtreqless 1$. Inverting,

$$\chi = \frac{1}{i} \tan i\psi = \frac{1}{i} \frac{\sin(x+iy)}{\cos(x+iy)} = \frac{\sinh y \cosh y - i \sin x \cos x}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$$

(A-18)

where $i\psi \equiv x + iy$. The condition that χ be real implies that either

$$x = n\pi \quad \text{or} \quad x = (2n-1)\frac{\pi}{2} \qquad \text{(A-19)}$$

which in turn gives

$$\psi = \tanh^{-1}\chi - in\pi \quad \text{or} \quad \psi = \coth^{-1}\chi - i(2n-1)\frac{\pi}{2}. \qquad \text{(A-20)}$$

In either case the imaginary constant may be included as part of the constant of integration so that $H(U)$ remains real.

Using the appropriate values of ϕ, α^2, and $\bar{k}$, the formula (A-16) for $\Pi(\phi,\alpha^2,\bar{k})$ becomes for $U < \sqrt{\beta/3}$

$$\Pi(\phi,\alpha^2,\bar{k}) = F(\phi,\bar{k}) - \Pi(\phi,\tfrac{1}{4},\bar{k}) + \frac{2}{\sqrt{3}}\sqrt{\beta+1}\,\tanh^{-1}\left[\frac{3U}{\beta}\sqrt{\frac{\beta+U^2}{4+\beta-3U^2}}\right]$$

(A-21)

and Eq. (A-14), completely in terms of tabulated functions, is

$$\mp H(U) = \frac{3\sqrt{3}}{\sqrt{\beta+1}} F(\phi,\bar{k}) + \sqrt{3(\beta+1)}\,E(\phi,\bar{k}) + \frac{3\sqrt{3}(\beta-3)}{4\sqrt{\beta+1}}\,\Pi(\phi,\tfrac{1}{4},\bar{k})$$

$$-\frac{\sqrt{3}}{2}(\beta-3)\tanh^{-1}\left[\frac{3U}{\beta}\sqrt{\frac{3+U^2}{4+\beta-3U^2}}\right] \qquad \text{(A-22)}$$

where

$$\left.\begin{aligned} \sin\phi &= \sqrt{\frac{4(\beta+1)U^2}{(\beta+4)(\beta+U^2)}} \\ \\ \bar{k}^2 &= \frac{\beta+4}{4(\beta+1)} \end{aligned}\right\} . \qquad \text{(A-23)}$$

For $U > \sqrt{\beta/3}$, Eqs. (A-21) and (A-22) must be modified by replacing $\tanh^{-1}$ by $\coth^{-1}$. The function $H(U)$ is plotted in Fig. 10 for $\beta = 5 > 3$ in the range $0 \leq U \leq 1$, and for $\beta = 2 < 3$ in the range $0 \leq U < \sqrt{2/3}$. No plot for $U > \sqrt{\beta/3}$ is shown because this case does not actually occur in the solutions constructed in Sec. IV. An infinity arises at $U = \sqrt{2/3}$ in Fig. 10, for the case $\beta = 2 < 3$, because the argument of $\tanh^{-1}$ in Eq. (A-22) approaches unity and the function becomes infinite. In the case $\beta > 3$, a similar situation exists, but on the branch $U > \sqrt{\beta/3}$ which is not plotted.

For the special case $\beta = 3$, Eq. (A-22) becomes

$$\mp H(U) = \frac{3}{2}\sqrt{3}\, F\left(\phi, \frac{\sqrt{7}}{4}\right) + 2\sqrt{3}\, E\left(\phi, \frac{\sqrt{7}}{4}\right) \qquad \text{(A-24)}$$

which is always finite.

Figure 10 shows typical plots of $H(U)$ in the region where Eq. (A-22) applies. For a considerable part of the range $H(U)$ is nearly linear , so that the first term of the power series for $H(U)$

$$\mp H(U) = 2U\sqrt{\frac{3}{\beta}(\beta+4)} \qquad \text{(A-25)}$$

may be used as an approximation.

2. EVALUATION OF $\dot{u}$

Differentiation of Eq. (3-14) yields

$$\frac{s'_x}{k} = \pm \frac{2}{\sqrt{3}} \sqrt{\frac{\beta + U^2}{4+\beta-3U^2}} \left[\frac{\beta(4+\beta) - 6\beta U^2 - 3U^4}{(\beta + U^2)^2}\right] \quad \text{(A-26)}$$

which, together with Eq. (3-19b), gives the derivative of $\dot{u}$

$$\sqrt{\frac{\rho G}{k^2}}\ \dot{u}' = \pm \sqrt{\frac{3(\beta+U^2)}{4+\beta-3U^2}} \left[\frac{\beta(4+\beta) - 6\beta U^2 - 3U^4}{(\beta-3U^2)(\beta+U^2)^2}\right] \quad \text{(A-27)}$$

and by integration

$$\pm \sqrt{\frac{\rho G}{k^2}}\ \dot{u}(U) = H_u(U) \pm \text{const.} \quad \text{(A-28)}$$

where

$$H_u(U) \equiv \int \sqrt{\frac{3(\beta+U^2)}{4+\beta-3U^2}} \left[\frac{\beta(4+\beta) - 6\beta U^2 - 3U^4}{(\beta-3U^2)(\beta+U^2)^2}\right] 2U dU. \quad \text{(A-29)}$$

Introduction of the new variable

$$z = \sqrt{\frac{3(\beta+U^2)}{4+\beta-3U^2}} \quad \text{(A-30)}$$

gives

$$H_u(U) = \frac{1}{2}\int \frac{3\beta + 6\beta z^2 - (4+\beta)z^4}{z^2(\beta-z^2)(1+z^2)}\, dz\ . \quad \text{(A-31)}$$

When $U < \sqrt{\beta/3}$, i.e., $z < \sqrt{\beta}$, the integral becomes

$$H_u(U) = 2 \tan^{-1} z - \frac{\beta-3}{2\sqrt{3}} \tanh^{-1}\frac{z}{\sqrt{\beta}} - \frac{3}{2z}\ . \quad \text{(A-32)}$$

Reintroducing the substitution (A-30) gives

$$H_u(U) = 2\tan^{-1}\sqrt{\frac{3(\beta+U^2)}{4+\beta-3U^2}} - \frac{(\beta-3)}{2\sqrt{\beta}}\tanh^{-1}\sqrt{\frac{3(\beta+U^2)}{\beta(4+\beta-3U^2)}}$$

$$- \frac{\sqrt{3}}{2}\sqrt{\frac{4+\beta-3U^2}{\beta+U^2}} \quad . \qquad \text{(A-33)}$$

Expressing the constant in Eq. (A-28) by the value of $\dot{u}$ at U_2 yields finally

$$\pm\sqrt{\frac{\rho G}{k^2}}\ \dot{u}(U) = \pm\sqrt{\frac{\rho G}{k^2}}\ \dot{u}(U_2) + H_u(U) - H_u(U_2) \qquad \text{(A-34)}$$

where the sign of $\dot{u}$ corresponds to that of s_x .

It should be noted that the argument of $\tanh^{-1}$ approaches unity as $U \to \sqrt{\beta/3}$. Thus $H_u(U)$ behaves similar to $H(U)$ and becomes infinite at $U = \sqrt{\beta/3}$, unless $\beta = 3$.

3. EVALUATION OF $\dot{v}$

Equation (3-19d) gives $\dot{v}'$ in terms of s_x, τ, and s_x' . Combining Eqs. (3-14), (3-15), and (A-26), one obtains

$$\sqrt{\frac{\rho G}{k^2}}\ \dot{v}' = \pm\sqrt{\frac{\beta + U^2}{(1-U^2)(\beta-3U^2)}}\left[\frac{\beta(4+\beta) - 6\beta U^2 - 3U^4}{(\beta+U^2)^2}\right] \qquad \text{(A-35)}$$

and by integration

$$\pm\sqrt{\frac{\rho G}{k^2}}\ \dot{v}(U) = H_v(U) \pm \text{const.} \qquad \text{(A-36)}$$

where

$$H_v(U) \equiv \int_0^U \sqrt{\frac{\beta + \xi^2}{(1-\xi^2)(\beta-3\xi^2)}}\left[\frac{\beta(4+\beta) - 6\beta\xi^2 - 3\xi^4}{(\beta+\xi^2)^2}\right] d\xi \qquad \text{(A-37)}$$

Using the value of $\dot{v}$ at the boundary of the region to evaluate the constant,

$$\pm\sqrt{\frac{\rho G}{k^2}}\ \dot{v}(U) = \pm\sqrt{\frac{\rho G}{k^2}}\ \dot{v}(U_2) + H_v(U) - H_v(U_2) \qquad \text{(A-38)}$$

where the sign of $\dot{v}$ is chosen to correspond to that of τ. Making the substitution $\eta = \xi^2$ in the integral gives

$$H_v(U) = \frac{1}{2}\int_0^{U^2} \sqrt{\frac{\beta+\eta}{(1-\eta)(\beta-3\eta)\eta}}\left[\frac{\beta(4+\beta) - 6\beta\eta - 3\eta^2}{(\beta+\eta)^2}\right] d\eta$$

$$= \frac{1}{2\sqrt{3}}\int_0^{U^2} \frac{\dfrac{\beta(4+\beta) - 6\beta\eta - 3\eta^2}{\beta+\eta}}{\sqrt{(\frac{\beta}{3} - \eta)(1-\eta)\eta(\eta+\beta)}}\, d\eta\ . \qquad \text{(A-39)}$$

The last expression may be compared with No. 254.41 in [6],

$$\int_c^y \frac{R(t)dt}{\sqrt{(a-t)(b-t)(t-c)(t-d)}} = g\int_0^{\hat{u}_1} R\left[\frac{c-\hat{\alpha}^2 d\ \mathrm{sn}^2\hat{u}}{1-\hat{\alpha}^2\mathrm{sn}^2\hat{u}}\right] d\hat{u} \quad \text{(A-40)}$$

where $R(t)$ is any rational function, where

$$a > b \geq y > c > d \qquad \text{(A-41)}$$

and

$$\mathrm{sn}^2\hat{u} = \frac{(b-d)(t-c)}{(b-c)(t-d)} \qquad \text{(A-42a)}$$

$$\hat{k}^2 = \frac{(b-c)(a-d)}{(a-c)(b-d)} \qquad \text{(A-42b)}$$

$$\hat{g} = \frac{2}{\sqrt{(a-c)(b-d)}} \qquad \text{(A-42c)}$$

$$0 < \hat{\alpha}^2 = \frac{b-c}{b-d} < \hat{k}^2 \qquad \text{(A-42d)}$$

$$\mathrm{sn}\hat{u}_1 = \sin\hat{\phi} = \sqrt{\frac{(b-d)(y-c)}{(b-c)(y-d)}} \ . \qquad \text{(A-42e)}$$

The cases $\beta > 3$, $\beta < 3$, and $\beta = 3$ must be treated separately.

Case 3.a When $\beta > 3$,

$$a = \frac{\beta}{3}\ , \quad b = 1\ , \quad c = 0\ , \quad d = -\beta\ , \quad y = U^2 \qquad \text{(A-43)}$$

satisfies the condition (A-41) for $0 < U \leq 1$, and Eqs. (A-42) give

$$\hat{k}^2 = \frac{4}{\beta+1}\ , \quad \hat{\alpha}^2 = \frac{1}{\beta+1}\ , \quad \hat{g} = \frac{2\sqrt{3}}{\sqrt{\beta(\beta+1)}} \qquad \text{(A-44)}$$

and

$$\sin\hat{\phi} = \mathrm{sn}\hat{u}_1 = U\sqrt{\frac{\beta+1}{\beta+U^2}} \ . \qquad \text{(A-45)}$$

The argument of R which appears on the right side of Eq. (A-40) is

$$\frac{\beta}{\beta+1}\ \frac{\mathrm{sn}^2\hat{u}}{\left(1-\frac{1}{\beta+1}\mathrm{sn}^2\hat{u}\right)}$$

and the integral (A-39) becomes

$$H_V(U) = \frac{1}{\sqrt{\beta(\beta+1)}}\int_0^{\hat{u}_1} \frac{4 + \ldots - 8\mathrm{sn}^2\hat{u} + \frac{4}{\beta+1}\mathrm{sn}^4\hat{u}}{1-\frac{1}{\beta+1}\mathrm{sn}^2\hat{u}}\, d\hat{u}. \qquad \text{(A-46)}$$

The use of the identity $dn^2\hat{u} \equiv 1 - \hat{k}^2 sn^2\hat{u}$, and Nos. 336.01, 337.01, 362.11 in [6], together with Eq. (A-44), results in the final form of the integral

$$H_V(U) = 3\sqrt{\frac{\beta+1}{\beta}}\; F(\hat{\phi},\hat{k}) + \sqrt{\frac{\beta+1}{\beta}}\; E(\hat{\phi},\hat{k}) - 3\sqrt{\frac{\beta}{\beta+1}}\; \Pi(\hat{\phi},\hat{\alpha}^2,\hat{k}) \tag{A-47}$$

where $\hat{k}$, $\hat{\alpha}$, and $\hat{\phi}$ are given by Eqs. (A-44) and (A-45). (In contrast to the situation encountered when determining $H(U)$, $\hat{\alpha}^2$ being here less than unity, the function Π is tabulated.)

<u>Case 3.b</u> When $\beta < 3$, the inequality (A-41) is satisfied in the range $0 < U \leq \sqrt{\beta/3}$ by

$$a = 1\ , \quad b = \frac{\beta}{3}\ , \quad c = 0\ , \quad d = -\beta\ , \quad y = U^2\ . \tag{A-48}$$

Eqs. (A-42) give

$$\hat{k}^2 = \frac{\beta+1}{4}\ , \quad \hat{\alpha}^2 = \frac{1}{4}\ , \quad g = \sqrt{\frac{3}{\beta}} \tag{A-49}$$

and

$$\sin\hat{\phi} = sn\,\hat{u}_1 = \frac{2U}{\sqrt{\beta+U^2}}\ . \tag{A-50}$$

The argument of R on the right-hand-side of Eq. (A-40) is

$$\frac{\beta}{4}\;\frac{sn^2\hat{u}}{(1-\frac{1}{4}\,sn^2\hat{u})}$$

and the integral (A-39) becomes

$$H_V(U) = \frac{1}{2\sqrt{\beta}} \int_0^{\hat{u}_1} \frac{4+\beta - (2\beta+1)\,sn^2\hat{u} - sn^2\hat{u}\; dn^2\hat{u}}{1 - \frac{1}{4}\, sn^2\hat{u}}\, d\hat{u}\,. \tag{A-51}$$

Using Nos. 336.01, 337.01, and 362.11 in [6] gives

$$H_V(U) = 2\sqrt{\beta}\, F(\hat{\phi},\hat{k}) + \frac{2}{\sqrt{\beta}}\, E(\hat{\phi},\hat{k}) - \frac{3}{2}\sqrt{\ }\; \Pi(\hat{\ },\tfrac{1}{4},\hat{k}) \tag{A-52}$$

where $\hat{k}$ and $\hat{\phi}$ are given by Eqs. (A-49) and (A-50), respectively.

Case 3.c When $\beta = 3$, Eq. (A-39) reduces to

$$H_V(U) = \frac{\sqrt{3}}{2} \int_0^{U^2} \frac{(7+\eta)\,d\eta}{(3+\eta)\sqrt{\eta(3+\eta)}} \tag{A-53}$$

which may be evaluated by using the substitution

$$z = \sqrt{\frac{\eta}{3+\eta}}\ .$$

The final result is

$$H_V(U) = \sqrt{3}\, \tanh^{-1} \frac{U}{\sqrt{3+U^2}} + \frac{4}{\sqrt{3}}\, \frac{U}{\sqrt{3+U^2}}\ . \tag{A-54}$$

The same result could also be obtained by substituting $\beta = 3$ into Eqs. (A-44) and (A-47), or Eqs. (A-49) and (A-52). For the resulting value of the modulus, $\hat{k} = 1$, the elliptic integrals reduce to simpler functions.

APPENDIX B

USE OF THE METHOD OF CHARACTERISTICS WHEN THE APPLIED SURFACE STRESSES ARE TIME DEPENDENT

In Sec. II, the basic partial differential equations governing propagation of plane waves in both dissipative and non-dissipative regions were obtained. In this Appendix it will be demonstrated that the characteristic velocities, V_i, of these differential equations may be obtained from Eqs. (3-14, 15) relating s_x, τ and U, but that difficulties arise in the application of the method of characteristics.

1. DISSIPATIVE REGIONS

Equations (2-15), (2-17), and Eq. (2-19) differentiated with respect to time, using also Eq. (3-3), give the quasi-linear set of differential equations

$$\begin{bmatrix} 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} s_x \\ J_1 \\ \dot{u} \\ \tau \\ \dot{v} \\ \Lambda \end{bmatrix} + \begin{bmatrix} \frac{1}{2G} & 0 & 0 & 0 & 0 & \frac{s_x}{G} \\ 0 & \frac{1}{\beta G} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G} & 0 & \frac{\tau}{G} \\ 0 & 0 & 0 & 0 & -\rho & 0 \\ \frac{3}{4}s_x & 0 & 0 & \tau & 0 & 0 \end{bmatrix} \frac{\partial}{\partial t} \begin{bmatrix} s_x \\ J_1 \\ \dot{u} \\ \tau \\ \dot{v} \\ \Lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{B-1}$$

Using matrix notation, Eq. (B-1) becomes

$$A' u_x + B' u_t = 0 \tag{B-2}$$

where u is the column matrix of the unknowns, and $|B'| \neq 0$. Premultiplying Eq. (B-2) by B'^{-1} gives

$$A u_x + u_t = 0 \tag{B-3}$$

where

$$A = B'^{-1} A' \ . \tag{B-4}$$

Using the procedure in [11], the characteristic velocities V are obtained from

$$|A - VI| = 0 \tag{B-5}$$

where I is the unit matrix, and

$$A = \begin{bmatrix} 0 & 0 & -\frac{4G\tau^2}{3k^2} & 0 & \frac{Gs_x\tau}{k^2} & 0 \\ 0 & 0 & -\rho G & 0 & 0 & 0 \\ -\frac{1}{\rho} & -\frac{1}{3\rho} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{Gs_x\tau}{k^2} & 0 & -\frac{3Gs_x^2}{4k^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{\rho} & 0 & 0 \\ 0 & 0 & -\frac{Gs_x}{2k^2} & 0 & -\frac{G\tau}{2k^2} & 0 \end{bmatrix} . \tag{B-6}$$

Making the substitution

$$V = \pm\sqrt{\frac{G}{\rho}}\ U \qquad \text{(B-7)}$$

the condition (B-5) gives

$$U^2[(1-U^2)(\beta-3U^2) - (\frac{\tau}{k})^2(\beta+U^2)] = 0 \ . \qquad \text{(B-8)}$$

This equation is identically satisfied by Eq. (3-15) so that the values U as functions of τ (or s_x) define the non-vanishing characteristic velocities.

Solution of Eq. (B-8) for U^2 gives, in addition to the root $U^2 = 0$, always two different positive roots U^2 so that there are four characteristic velocities, $V_1=-V_2$, $V_3 = -V_4$, in addition to the degenerate double value $V_o = 0$.

After appropriate manipulation, the compatibility equations along the four non-vanishing characteristics become

$$\Delta s_x + \frac{1}{3}\Delta J_1 - \rho V_i \Delta\dot{u} + \frac{k^2}{s_x\tau}\left[\frac{4\tau^2}{3k^2} + \frac{\beta}{3} - \frac{\rho V_i^2}{G}\right]\Delta\tau$$

$$- \frac{\rho V_i k^2}{s_x\tau}\left[\frac{4\tau^2}{3k^2} + \frac{\beta}{3} - \frac{\rho V_i^2}{G}\right]\Delta\dot{v} = 0 \qquad \text{(B-9)}$$

where Δs_x , ΔJ_1 , etc., are the increments of the respective functions along the characteristics. The degenerate double value, $V_o = 0$, furnishes the relations

$$\frac{3}{4}\ s_x \Delta s_x + \tau\Delta\tau = 0 \qquad \text{(B-10)}$$

$$2\ s_x G\lambda = \frac{4}{3\beta}\Delta J_1 - \Delta s_x \ . \qquad \text{(B-11)}$$

The first of these equations can be recognized as the increment of the yield condition.

The five Eqs. (B-9, 10) permit the step by step evaluation of the five quantities s_x, J_1, $\dot{u}$, τ, and $\dot{v}$ within a dissipative region, while Eq. (B-11) provides the necessary check that $\Lambda > 0$, required in a dissipative region.

2. NON-DISSIPATIVE REGIONS

Starting with Eqs. (2-15) and (2-21), one finds the equation

$$V[V^2 - \frac{G}{\rho}(\frac{4+\beta}{3})][V^2 - \frac{G}{\rho}] = 0 \qquad \text{(B-12)}$$

for the characteristic velocities. There is one degenerate value

$$V_o = 0$$

and the four well-known values

$$V_{1,2} = \pm c_P \equiv \pm \sqrt{\frac{G}{\rho}(\frac{4+\beta}{3})} \qquad \text{(B-13)}$$

$$V_{3,4} = \pm c_S \equiv \pm \sqrt{\frac{G}{\rho}} \qquad \text{(B-14)}$$

where c_P and c_S are the velocities of P- and S-waves, respectively. The relations along the characteristics are

$$\text{For } V_o = 0 : \quad \Delta s_x - \frac{4}{3\rho}\Delta J_1 = 0 \qquad \text{(B-15)}$$

$$\text{For } V_{1,2} = \pm c_P : \quad \Delta s_x + \frac{1}{3}\Delta J_1 \mp \rho c_P \Delta\dot{u} = 0 \qquad \text{(B-16)}$$

$$\text{For } V_{2,3} = \pm c_S : \quad \Delta\tau \mp \rho c_S \Delta\dot{v} = 0 \quad . \qquad \text{(B-17)}$$

These relations are applicable, provided the increments of the stresses are such that the inequality (2-22) remains satisfied.

The appearance of the degenerate value $V_o = 0$ is due to the fact that the formulation, Eqs. (2-15) and (2-21), uses s_x and J_1 as separate variables, instead of the single unknown $\sigma_x = s_x + J_1/3$ sufficient for the solution of a purely elastic problem.

3. DISCONTINUITIES

Elastic discontinuities of well known nature may propagate with the velocity of P- or S-waves, $\pm c_P$, $\pm c_S$, respectively, while dissipative shock fronts, discussed in Subsec. III.B, have a velocity $\bar{c} = \pm \sqrt{K/\rho}$, and are restricted to locations where $\tau = 0$.

When constructing solutions involving discontinuities, analytical relations are required to follow the interaction of elastic and plastic fronts with adjoining continuous plastic, and elastic regions, respectively. There is no particular difficulty in deriving the appropriate relations. They were not derived here, since they are not required for the discussion which follows.

4. DIFFICULTIES IN THE DETERMINATION OF SOLUTIONS

The difficulties encountered are best illustrated by considering the Cauchy boundary value problem. The usual finite difference technique can be used in any location which is known to be entirely in either an elastic or a dissipative region, removed from an interface or discontinuity. Figure 11 illustrates a step from t_1 to $t_1 + \Delta t$. The values of all functions are known for $t = t_1$, i.e., along A-E. At point P, there are five unknowns, which may be obtained from the respective five equations along the characteristics, subject to check on the yield condition, or on $\lambda > 0$, as appropriate. There is no difficulty at a point where yield has not been reached, the elastic relations must certainly apply. However, disregarding discontinuities, consider a situation where the yield condition is satisfied for a portion of the range A-E, indicating that there might be an elastic-plastic boundary. To locate a point P on the boundary and the inclination α of the latter, Fig. 12-a, one can combine the conditions along those elastic and plastic characteristics which lie in the proper regions, but, depending on the value of α, there are a variety of possibilities. There are six unknown quantities, the inclination α, three stresses, and two velocities, while the yield condition and five or six characteristic conditions are available. Over-determination, Fig. 12-b, in case the seven conditions apply is no difficulty. It may simply mean that the particular location may not occur. There are, however, values of α as shown in Fig. 12-c where six conditions apply, one of which is an inequality, with a possibility of indeterminancy. Before proceeding with a numerical analysis, it is necessary to ascertain the existence and uniqueness of the solutions for the location of, and stresses at a boundary point P from the conditions available for all possible situations. If this can be achieved, the additional possibility of discontinuities, the velocities of propagation of which are known, should be no serious obstacle.

REFERENCES

1. J. W. Craggs, "The Propagation of Infinitesimal Plane Waves in Elastic-Plastic Materials," J. of the Mechanics and Physics of Solids, Vol. 5, pp. 115-24 (1957).

2. Progress in Solid Mechanics, Vol. I, Chap. 3, Vol. II, Chap. 4, North-Holland Pub. Co. (1960-61).

3. A. M. Freudenthal and H. Geiringer, "The Mathematical Theories of the Inelastic Continuum," Handbuch der Physik, Vol. VI, Springer-Verlag, p. 326 (1958).

4. R. D. Richtmyer, Difference Methods for Initial-Value Problems, Interscience Publishers, Inc., pp. 208-10 (1957).

5. J. Mandel, "Ondes Plastiques Dans un Milieu Indefini à Trois Dimensions," Journal de Mechanique, 1, pp. 3-30 (1962).

6. P. Byrd and M. Friedman, Handbook of Elliptic Integrals for Engineers and Physicists, Springer-Verlag (1954).

7. Pierce and Foster, A Short Table of Integrals, 4th ed., Ginn and Company.

8. H. Hancock, Elliptic Integrals, Dover Publications, Inc. (1917).

9. R. G. Selfridge and J. E. Maxfield, A Table of the Incomplete Elliptic Integral of the Third Kind, Dover Publications, Inc. (1958).

10. R. V. Churchill, Complex Variables and Applications, 2nd ed., McGraw-Hill Book Co., Inc. (1960).

11. R. Courant and D. Hilbert, Methods of Mathematical Physics, Vol. II, Interscience Publishers, pp. 424-5 (1962).

TABLE I

FOR ALL VALUES OF β PROVIDED $-\sigma_0 < -\sigma_1$ [FOR σ_1 , SEE EQ. (4 − 4)]			
U	$\frac{S_x}{k}$	$\frac{J_1}{k}$	$\frac{\tau}{k}$
$U > \sqrt{\frac{4+\beta}{3}} \equiv U_P$	0	0	0
$\sqrt{\frac{4+\beta}{3}} > U > 1$	$\frac{4}{4+\beta} \frac{\sigma_0}{k}$	$\frac{3\beta}{4+\beta} \frac{\sigma_0}{k}$	0
$1 > U \geq 0$	$\frac{4}{4+\beta} \frac{\sigma_0}{k}$	$\frac{3\beta}{4+\beta} \frac{\sigma_0}{k}$	$\frac{\tau_0}{k}$
NOTE: MATERIAL REMAINS ELASTIC THROUGHOUT			

TABLE II

FOR $\beta \geq 3$, AND $-\sigma_1 < -\sigma_0 < -\sigma_2$ [FOR σ_1 , σ_2 , SEE EQS.(4-4), (4-7)]

FOR $\beta < 3$, $-\sigma_0 > -\sigma_1$ AND $\tau_0 \neq 0$

U	$\frac{S_x}{k}$	$\frac{J_1}{k}$	$\frac{\tau}{k}$
$U > \sqrt{\frac{4+\beta}{3}} \equiv U_P$	0	0	0
$\sqrt{\frac{4+\beta}{3}} > U > 1$	$\frac{4}{4+\beta}\,\frac{\Delta\sigma_x}{k}$	$\frac{3\beta}{4+\beta}\,\frac{\Delta\sigma_x}{k}$	0
$1 > U \geq U_2$	$\frac{4}{4+\beta}\,\frac{\Delta\sigma_x}{k}$	$\frac{3\beta}{4+\beta}\,\frac{\Delta\sigma_x}{k}$	$\frac{\Delta\tau}{k}$
$U_2 \geq U \geq U_1$	$-\sqrt{\frac{4U^2(4+\beta-3U^2)}{3(\beta+U^2)}}$	$\frac{3\beta}{4+\beta}\,\frac{\Delta\sigma_x}{k} + H(U) - H(U_2)$	$\sqrt{\frac{(1-U^2)(\beta-3U^2)}{\beta+U^2}}$
$U_1 \geq U \geq 0$	$-\frac{2}{\sqrt{3}}\sqrt{1-\left(\frac{\tau_0}{k}\right)^2}$	$3\left[\frac{\sigma_0}{k}+\frac{2}{\sqrt{3}}\sqrt{1-\left(\frac{\tau_0}{k}\right)^2}\right]$	$\frac{\tau_0}{k}$

SUMMARY OF BASIC RELATIONS

$$\frac{12}{(4+\beta)^2}\left(\frac{\Delta\sigma_x}{k}\right)^2 + \left(\frac{\Delta\tau}{k}\right)^2 = 1 \quad , \quad \frac{\tau_0}{k} = \sqrt{\frac{(1-U_1^2)(\beta-3U_1^2)}{\beta+U_1^2}} \quad , \quad \frac{\Delta\tau}{k} = \sqrt{\frac{(1-U_2^2)(\beta-3U_2^2)}{\beta+U_2^2}}$$

$$\frac{3\beta}{4+\beta}\,\frac{\Delta\sigma_x}{k} + H(U_1) - H(U_2) = 3\left[\frac{\sigma_0}{k}+\frac{2}{\sqrt{3}}\sqrt{1-\left(\frac{\tau_0}{k}\right)^2}\right]$$

[FOR $H(U)$, SEE EQ.(3-28)]

NOTE: MATERIAL BECOMES PLASTIC AT S-FRONT , $U_S = 1$

TABLE III

FOR $\beta \geq 3$, AND $-\sigma_0 > -\sigma_2$ [FOR σ_2, SEE EQ. (4-7)]

U	$\frac{s_x}{k}$	$\frac{J_1}{k}$	$\frac{\tau}{k}$
$U > \sqrt{\frac{4+\beta}{3}} \equiv U_P$	0	0	0
$\sqrt{\frac{4+\beta}{3}} > U > \sqrt{\frac{\beta}{3}}$	$-\frac{2}{\sqrt{3}}$	$-\frac{\sqrt{3}}{2}\beta$	0
$\sqrt{\frac{\beta}{3}},\ U \geq 1$	$-\frac{2}{\sqrt{3}}$	$-\frac{\sqrt{3}}{2}\beta + \frac{\Delta J_1}{k}$	0
$1 \geq U \geq U_1$	$-\sqrt{\frac{4U^2(4+\beta-3U^2)}{3(\beta+U^2)}}$	$-\frac{\sqrt{3}}{2}\beta + \frac{\Delta J_1}{k} + H(U) - H(1)$	$\sqrt{\frac{(1-U^2)(\beta-3U^2)}{\beta+U^2}}$
$U_1 \geq U \geq 0$	$-\frac{2}{\sqrt{3}}\sqrt{1-(\frac{\tau_0}{k})^2}$	$3\left[\frac{\sigma_0}{k} + \frac{2}{\sqrt{3}}\sqrt{1-(\frac{\tau_0}{k})^2}\right]$	$\frac{\tau_0}{k}$

SUMMARY OF BASIC RELATIONS

$$\frac{\Delta\sigma_x}{k} = -\frac{4+\beta}{2\sqrt{3}}, \quad \Delta\tau = 0, \quad \frac{\tau_0}{k} = \sqrt{\frac{(1-U_1^2)(\beta-3U_1^2)}{\beta+U_1^2}}$$

$$\frac{\Delta J_1}{k} = \frac{\sqrt{3}}{2}\beta + H(1) - H(U_1) + 3\left[\frac{\sigma_0}{k} + \frac{2}{\sqrt{3}}\sqrt{1-(\frac{\tau_0}{k})^2}\right],$$ [FOR $H(U)$, SEE EQ. (3-28)]

NOTE: MATERIAL BECOMES PLASTIC AT P-FRONT, $U_P = \sqrt{\frac{4+\beta}{3}}$

TABLE IV

FOR $\beta < 3$, $-\sigma_o > \frac{4+\beta}{2\sqrt{3}}$ AND $\tau_o \equiv 0$

U	$\frac{s_x}{k}$	$\frac{J_1}{k}$	$\frac{\tau}{k}$
$U > \sqrt{\frac{4+\beta}{3}} \equiv U_P$	0	0	0
$\sqrt{\frac{4+\beta}{3}} > U > \sqrt{\frac{\beta}{3}}$	$-\frac{2}{\sqrt{3}}$	$-\frac{\sqrt{3}}{2}\beta$	0
$\sqrt{\frac{\beta}{3}} > U \geq 0$	$-\frac{2}{\sqrt{3}}$	$3\left[\frac{\sigma_o}{k} + \frac{2}{\sqrt{3}}\right]$	0

NOTE : MATERIAL BECOMES PLASTIC AT P-FRONT , $U_P = \sqrt{\frac{4+\beta}{3}}$

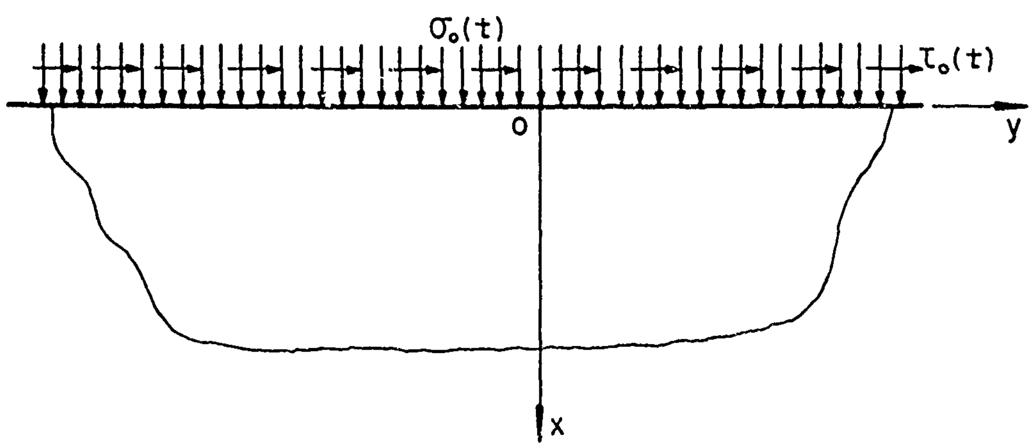


FIG. 1 HALF-SPACE SUBJECTED TO UNIFORM LOADS $\sigma_o(t)$ AND $\tau_o(t)$ AT THE SURFACE $X = 0$

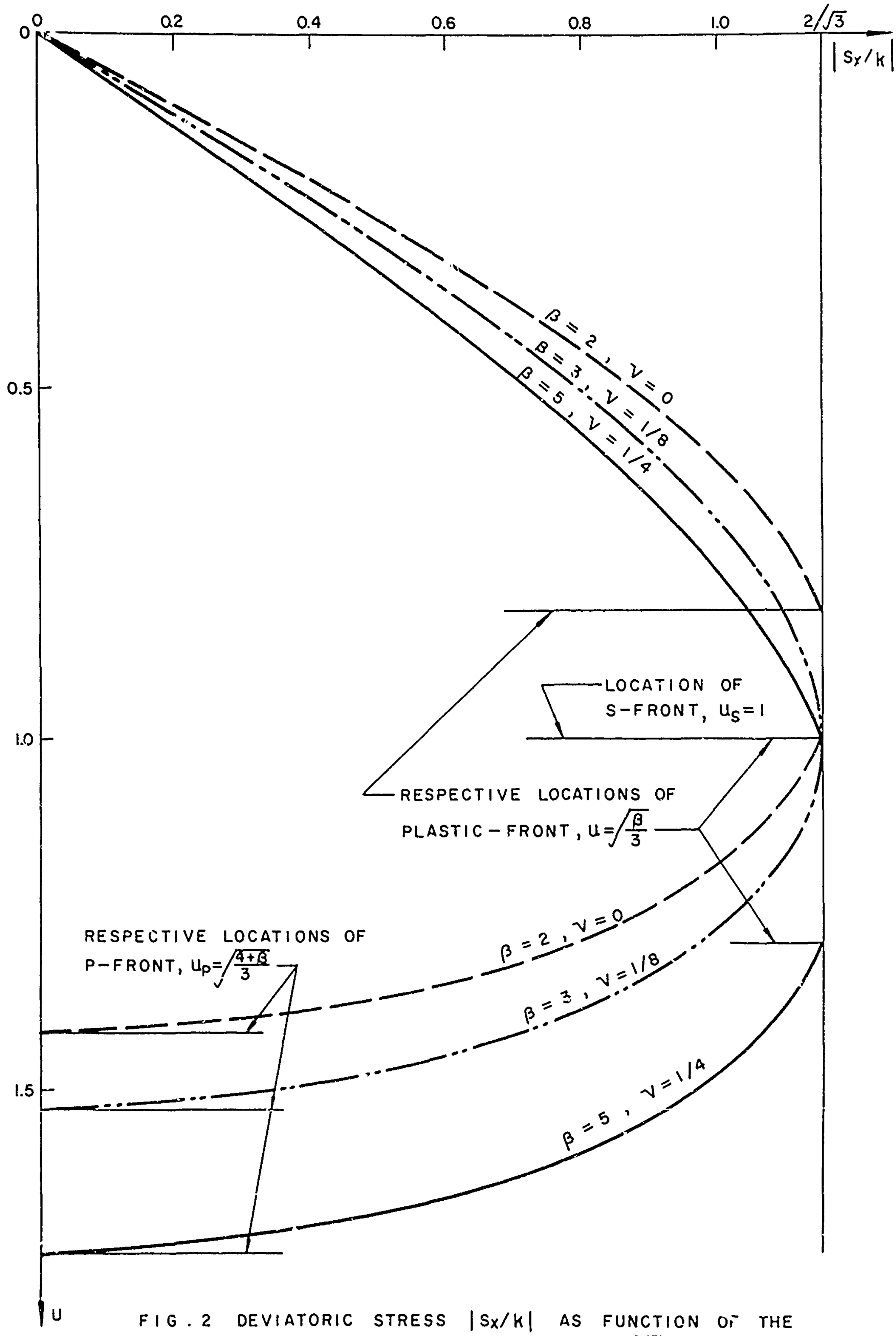


FIG. 2 DEVIATORIC STRESS $|S_x/k|$ AS FUNCTION OF THE NON-DIMENSIONAL VARIABLE $u = \sqrt{\frac{\rho}{G}}\,\frac{x}{t}$

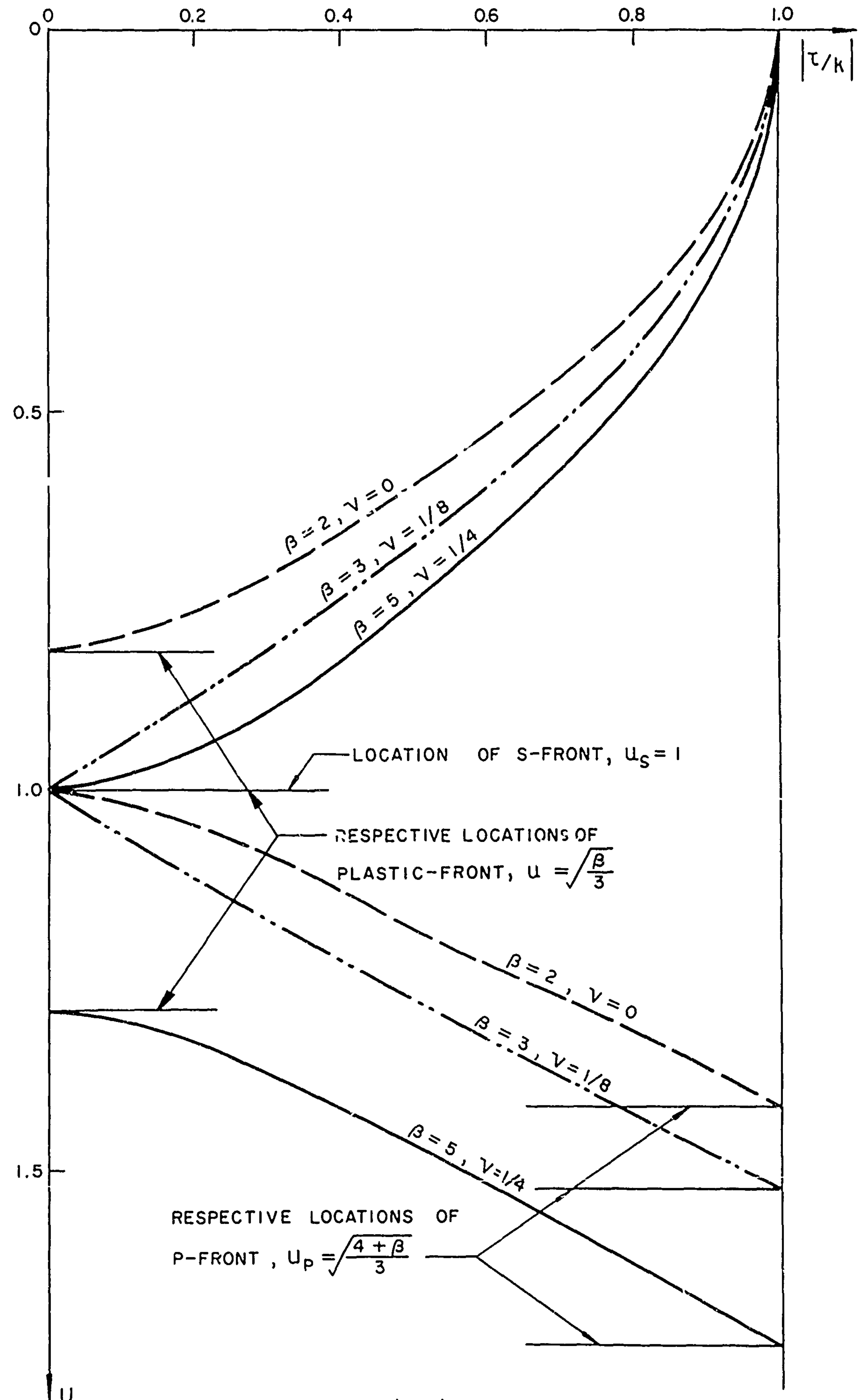


FIG. 3 SHEAR STRESS $|\tau/k|$ AS FUNCTION OF THE NON-DIMENSIONAL VARIABLE $u = \sqrt{\frac{\rho}{G}}\ \frac{x}{t}$

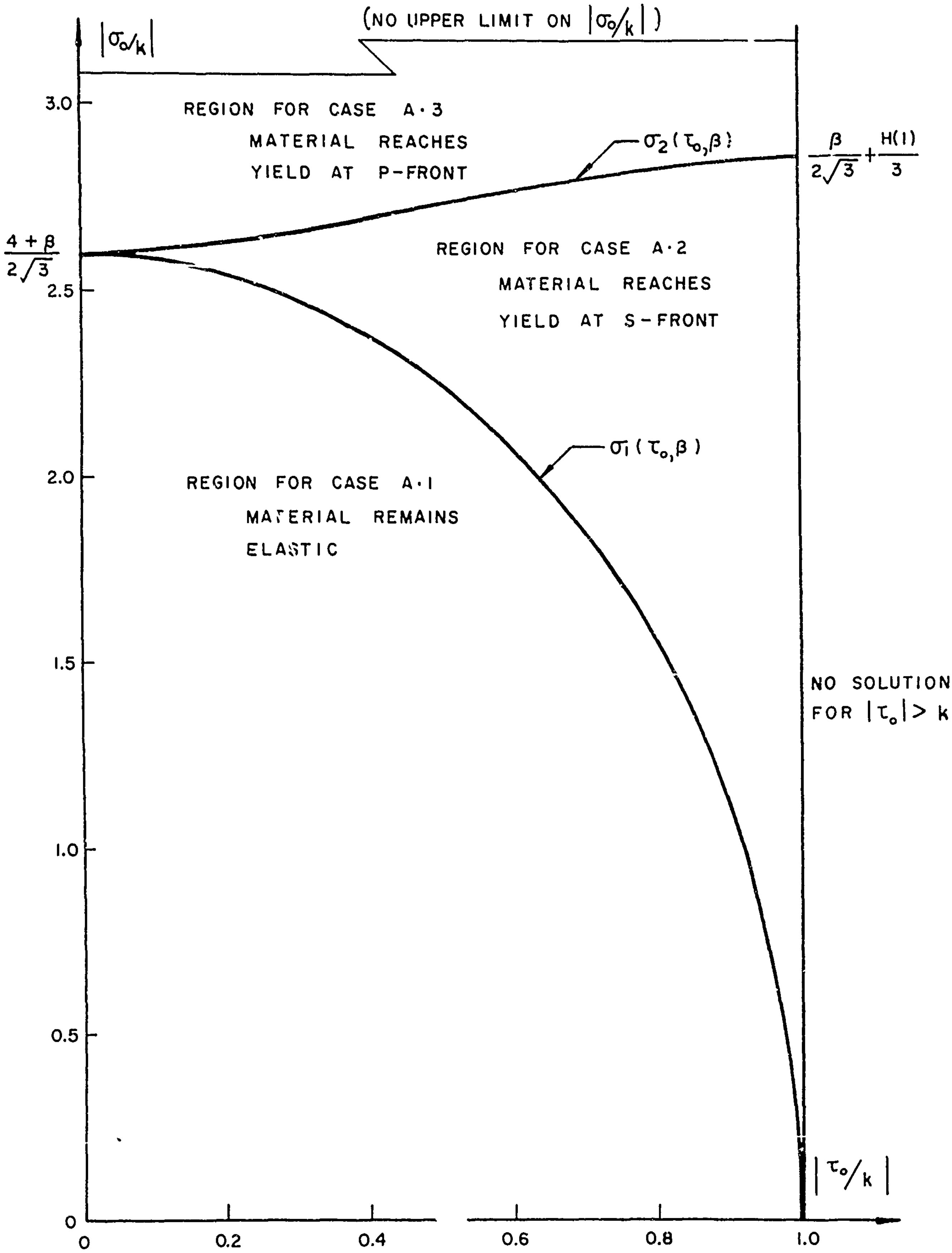


FIG. 4 DEPENDENCE OF TYPE OF SOLUTION ON SURFACE STRESSES σ_0, τ_0 WHEN $\beta \geq 3$ (DRAWN FOR $\beta = 5$)

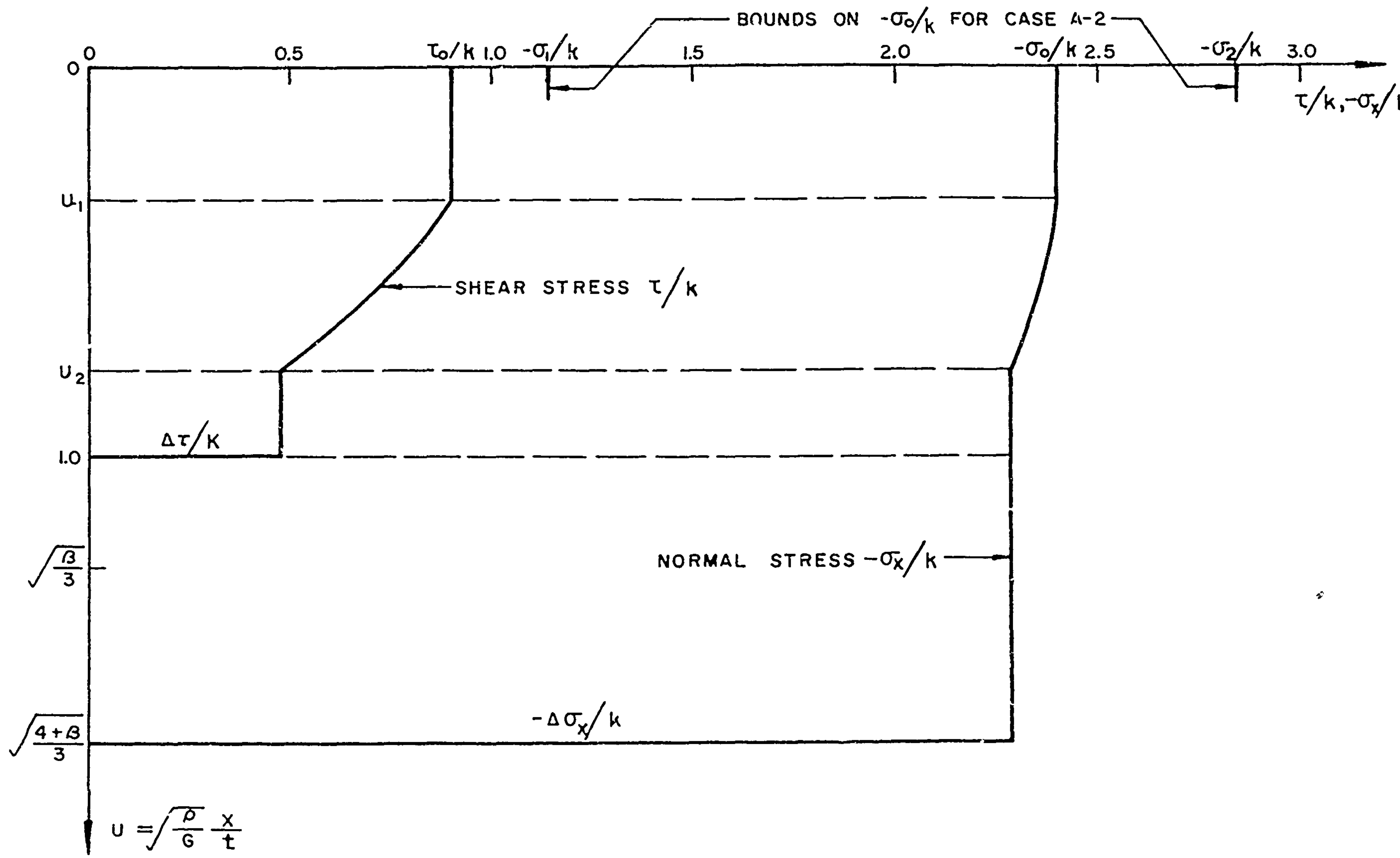
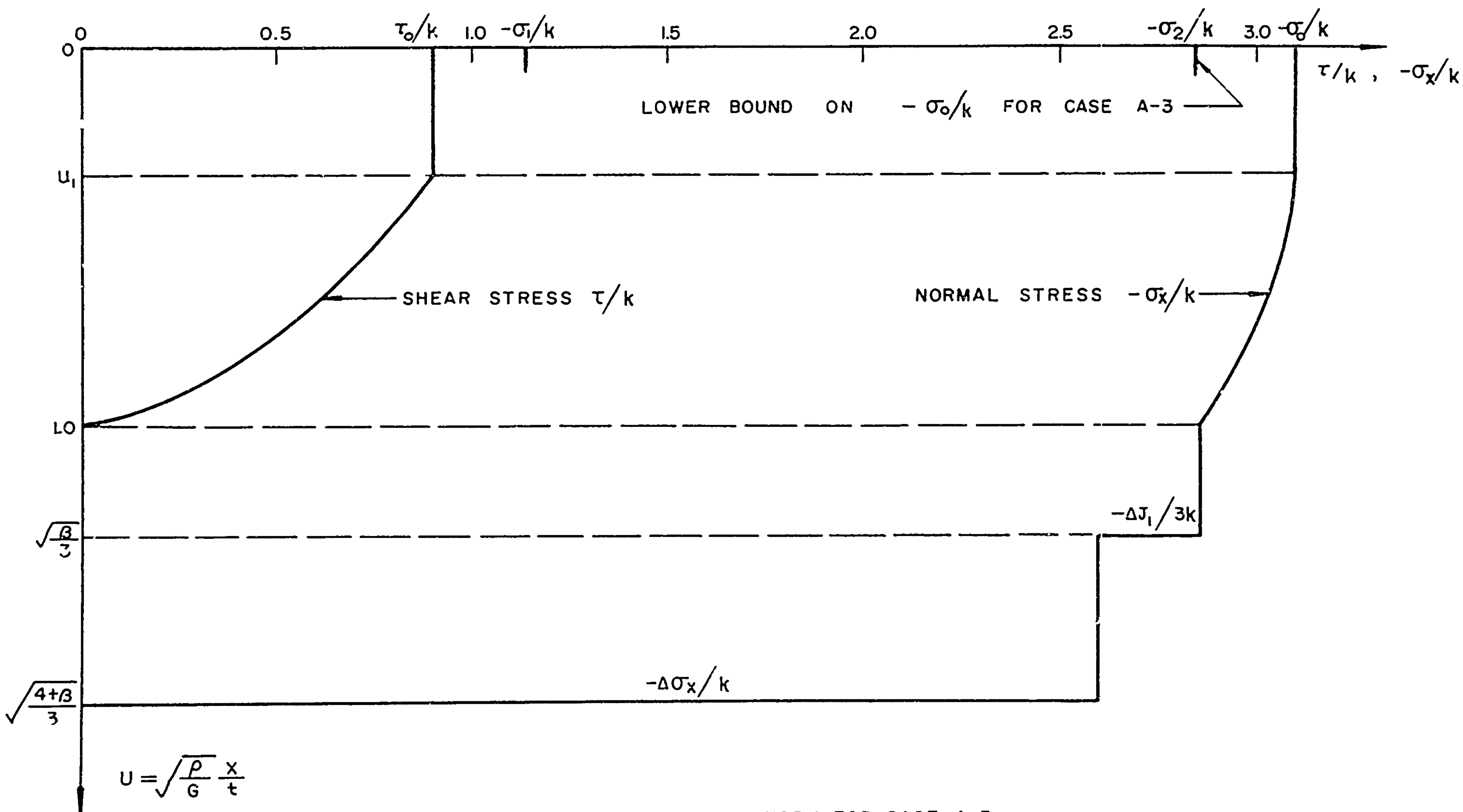


FIG. 5 TYPICAL STRESS HISTORY FOR CASE A.2

(DRAWN FOR $\beta = 5$, $\sigma_o = -2.4k$, $\tau_o = 0.9k$)



F I G . 6 TYPICAL STRESS HISTORY FOR CASE A·3

(DRAWN FOR $\beta = 5$, $\sigma_o = -3.1k$, $\tau_o = 0.9k$)

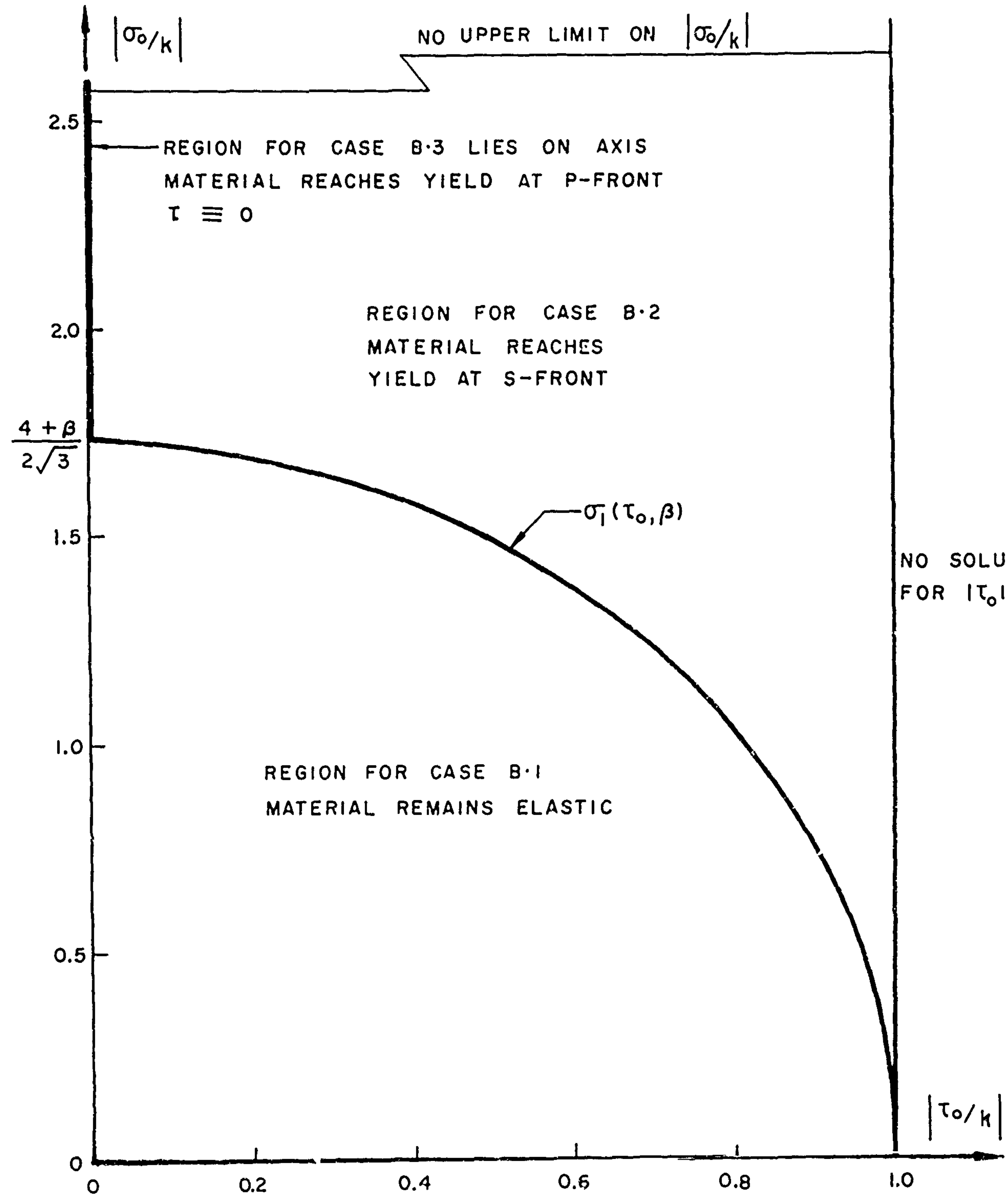


FIG. 7 DEPENDENDE OF TYPE OF SOLUTION ON SURFACE STRESSES σ_0, τ_0 WHEN $\beta < 3$ (DRAWN FOR $\beta = 2$)

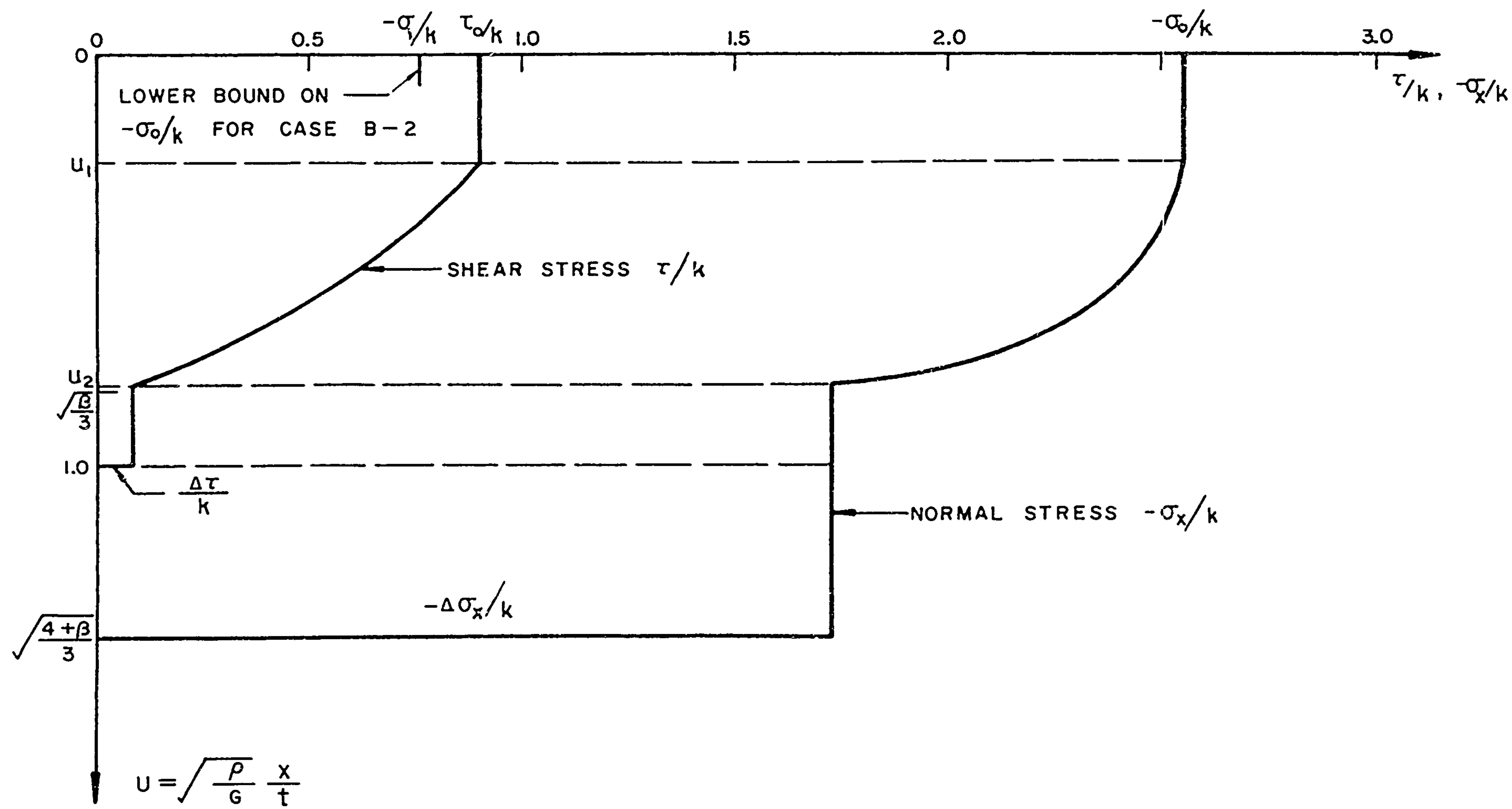


FIG. 8 TYPICAL STRESS HISTORY FOR CASE B·2

(DRAWN FOR $\beta=2$, $\sigma_0=-2.55k$, $\tau_0=0.9k$)

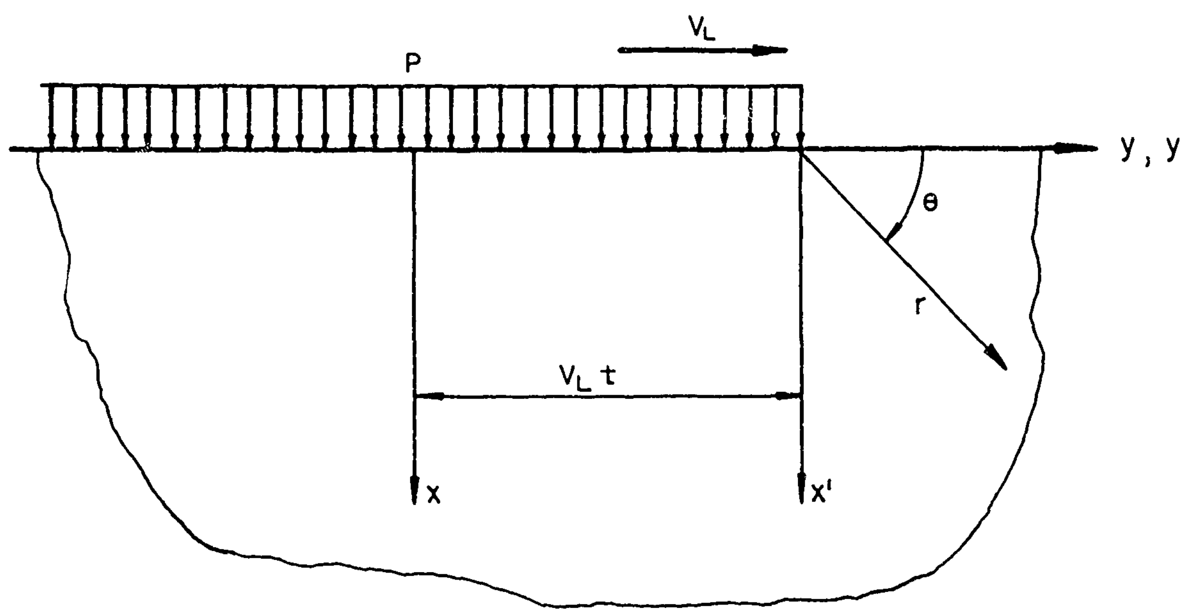


FIG. 9 HALF-SPACE LOADED BY MOVING STEP PRESSURE ON SURFACE

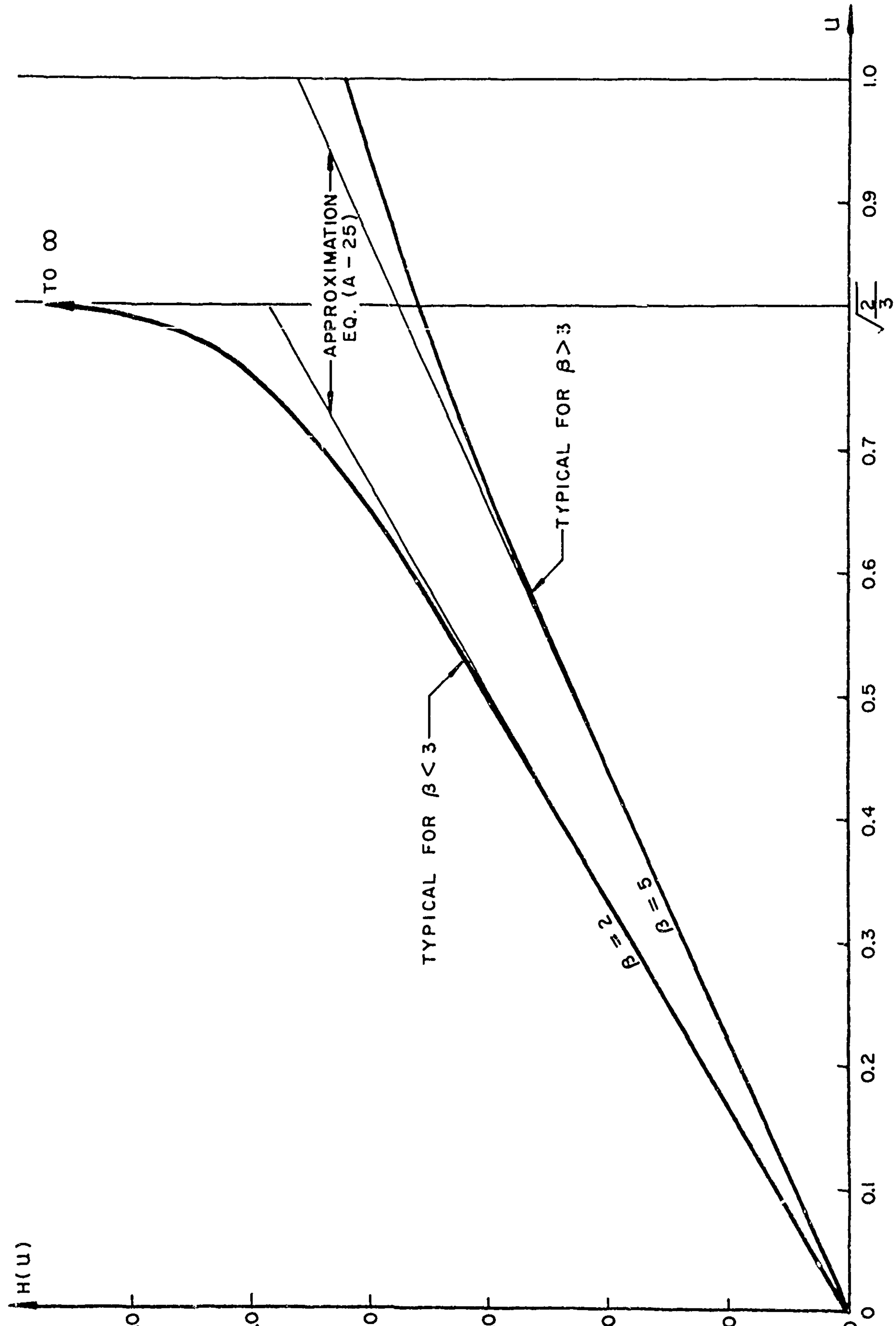


FIG. 10 TYPICAL PLOTS OF H(U)

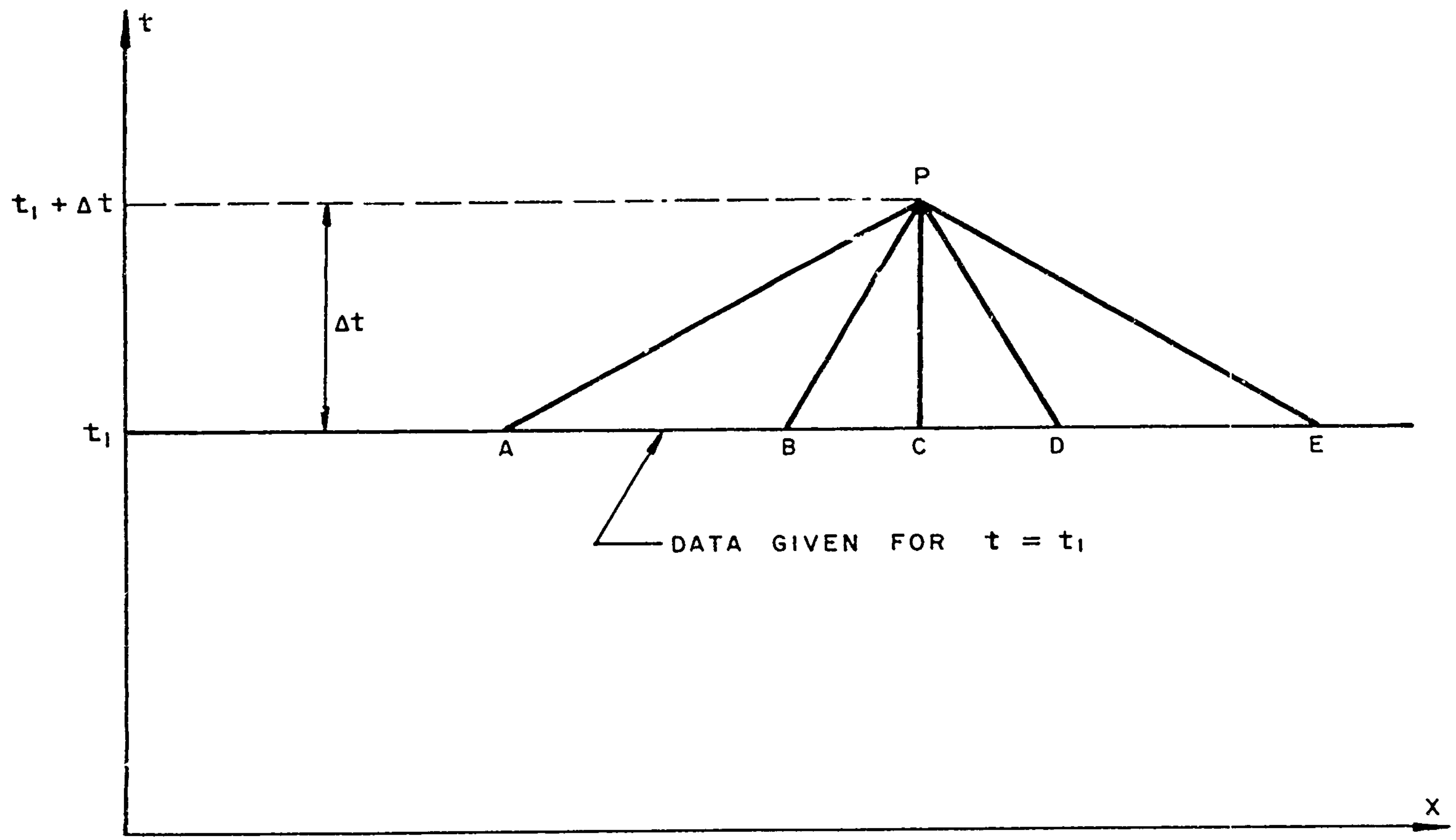


FIG. 11 SCHEME FOR SOLUTION OF CAUCHY PROBLEM BY METHOD OF CHARACTERISTICS

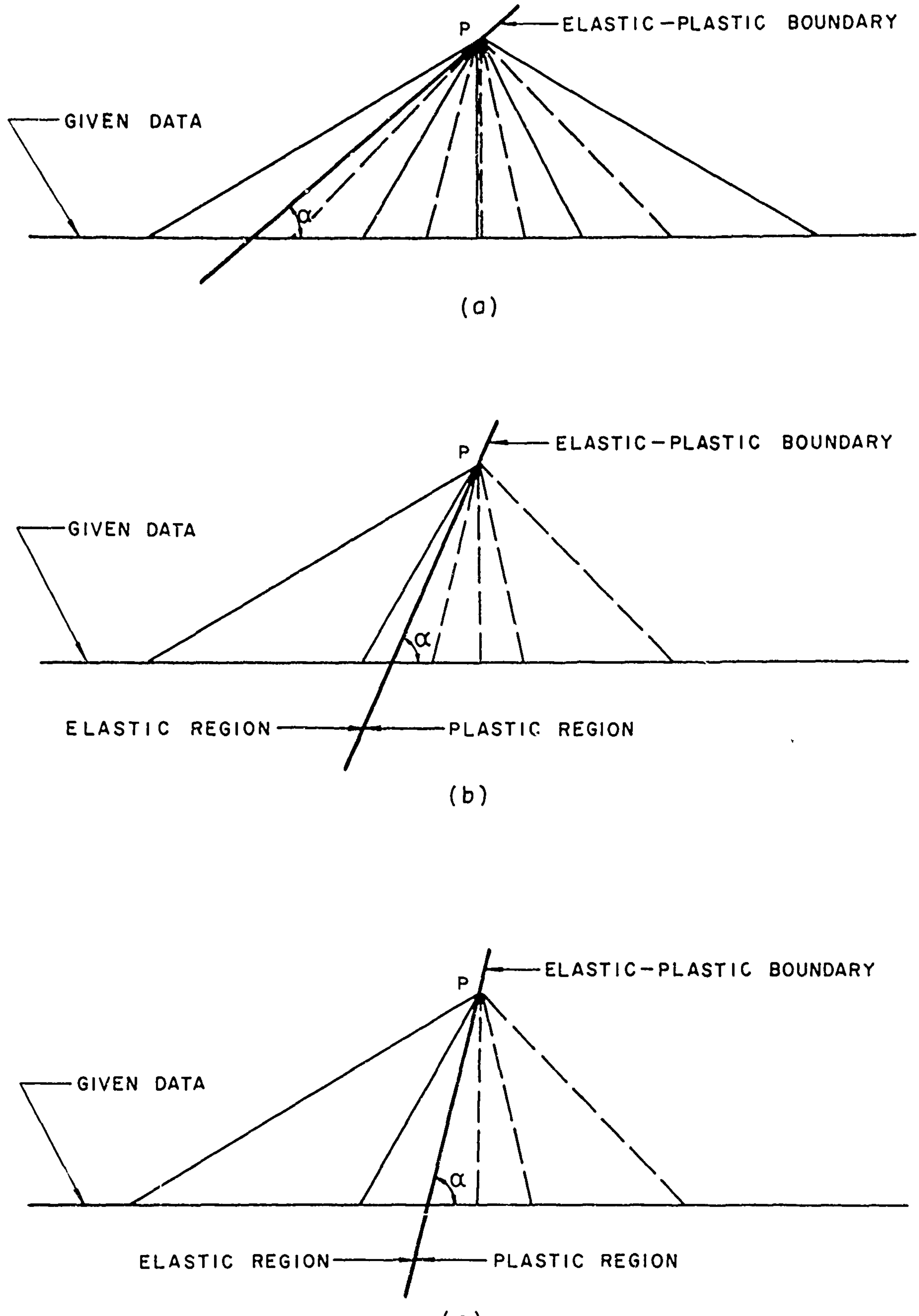


FIG. 12 CHARACTERISTIC SOLUTIONS AT ELASTIC-PLASTIC BOUNDARY

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